Asymptotic Safety of Gauge Theories and Gravity:

Adventures in large group dimensions

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree. I have read and understood the definition of *Plagiarism* as set out in the Examination and Assessment Handbook for Postgraduate Students under item 7. This dissertation is my own work, except where explicitly stated.

Parts of chapter 3 have been published as conference proceedings in [1].

Signature:

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Summary

Quantum field theories are considered fundamental provided they remain well-defined and predictive up to highest energies. Important such examples are known as asymptotic freedom and asymptotic safety where the high energy behaviour is controlled by a free or interacting fixed point under the renormalisation group, respectively. The focus of this thesis is the prospect of asymptotic safety for gauge theories and gravity. We are particularly interested in regimes where asymptotic safety arises at parametrically small coupling such as in large-N limits, where N relates to the degrees of freedom.

Specifically, in the first part, we investigate exact ultraviolet (UV) fixed points of recently discovered four-dimensional gauge Yukawa theories in the Veneziano limit of $SU(N)$ gauge theories coupled to matter. We include higher dimensional scalar self-interactions. Our main tools are perturbation theory in conjunction with non-perturbative "functional" renormalisation group (RG) techniques. It is established that classically irrelevant couplings take well-defined interacting fixed point values of their own, despite of their non-renormalisability within perturbation theory. We also establish vacuum stability, and show that higher order couplings remain parametrically irrelevant with near-Gaussian scaling exponents. Our results provide a crucial consistency check for exact asymptotic safety of weakly coupled gauge theories.

Secondly, we perform a large-N study for quantum Einstein gravity. The main novelty here is that the number of space-time dimensions D takes the role of the number of degrees of freedom. We then derive and analyse renormalisation group equations within a $1/D$ expansion, also comparing the so-called single and bimetric approximations and the gauge-fixing dependence of results. In either of the cases we find an asymptotically safe gravitational fixed point and a finite radius of convergence in the $1/D$ expansion. We discuss the consistency of our results in comparison with previous findings, and in the light of the asymptotic safety conjecture for gravity in four dimensions.

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I dedicate this work to my mother.

"(...) And science, you remember, is the study
of the nature and behaviour of the universe,
based on observation, experiment, and measurement,
and the formulation of laws to describe these facts.

The race continues. An early scientist
drew beasts upon the walls of caves
to show her children, now all fat on mushrooms
and on berries, what would be safe to hunt.

The men go running on after beasts.

The scientists walk more slowly, over to the brow of the hill
and down to the water's edge and past the place where the red clay runs.
They are carrying their babies in the slings they made,
freeing their hands to pick the mushrooms."

Neil Gaiman - Mushroom Hunters

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Chapter 1

Introduction

In this thesis, we attempt to address two of the biggest questions in (post-)modern physics.

1. After one of the most successful discoveries of the last century by CERN, the Higgs boson [2, 3], we know that the particle content of the standard model is complete. But it is not enough on its own to explain some of the physical phenomena, such as dark matter, baryon asymmetry, hierarchy problem, etc. If the particle content of the standard model is not sufficient, what is beyond the standard model of particles? Is there new physics? Are there more particles?

2. After more than a hundred years of Einstein's discovery of general relativity, we still do not know how gravity interacts with quantum particles. Is general relativity quantum in nature? Can gravity be described by a predictive quantum field theory, which has proved itself to be very successful for other theories such as the standard model?

We start addressing these questions with a simple philosophy: predictive quantum field theories must have measurable interactions. So if quantum gravity is a fundamental theory of nature, or if there are more fundamental particles beyond the standard model, we must be able to measure their interactions. The measurable strength of an interaction is the coupling constant. Relativistic quantum field theory provides us with a great opportunity that says that the coupling constants *run*. This is the opportunity to probe the high energy physics by relating it to low energy physics. The *running couplings* are by definition scale-dependent. This scale can be considered either the length or the energy scale that we are looking into these interactions.

We say that to be able to measure the interactions the coupling constants should go to *fixed points* and, when integrated along their flow from one scale to another, we should be able to do this without fixing an infinite number of parameters. So we want fixed points and a finite number of independent parameters to fix. Such theories are said to be asymptotically safe.

Fixed points are powerful tools to investigate fundamental and predictive quantum field theories up to high energies. At high energies, theories can reach a free or an interacting fixed point. Asymptotic freedom is a well-known example of the former case [4, 5]. Here we turn to asymptotic safety, the latter case, where the couplings reach an interacting fixed point at high energies.

The conjecture of asymptotic safety was introduced and investigated in reference [6], with the motivation of finding a well-defined fundamental quantum gravity theory. It has been a very active research area since the introduction of new mathematical tools such as functional renormalisation group theory [7, 8, 9]. This is used to derive the flow of quantum gravity in [10, 11]. The optimised cut-off is introduced that enables the solution of functional integrals in [12, 13]. β-functions of quantum gravity were computed using this cut-off in [14]. Some reviews of asymptotic safety of quantum gravity are given in [15, 16, 17, 18, 19, 20, 21]. Asymptotic safety of gauge theories have recently been investigated from the perturbation theory point of view [22, 23].

Motivation for the Large-N Limit: We not only look into asymptotic safety, but we do so by taking a limit where the group degrees of freedom is very large. So we make the theories in question very dynamic by increasing the degrees of freedom. In quantum field theory, a very successful non-perturbative expansion scheme is given by the large-N expansion, where N is related to the number of degrees of freedom of a gauge group. For example for an $SU(N)$ gauge group, degrees of freedom would be $N^2 - 1$. A large-N expansion involves deriving expressions for the physical properties of the system as a $1/N$ expansion by redefining the coupling constants. In $O(N)$ ϕ^4 theory, for example, it can be computed that the four-point vertices are suppressed by $1/N$ [24]. This, therefore, leads to a simplification in the perturbative expansion. Again, in statistical physics, the large-N limit often leads to an important simplification as it corresponds to a mean field approximation where quantum fluctuations are suppressed by $1/N$. In mean field approximation, by definition the fluctuations are suppressed. Therefore with a redefinition of the field, it can be shown that large-N limits leads to simplifications. A review of the applications of large-N to $O(N)$ invariant theories in relation to mean field approximation can be found in [24]. A systematic expansion in $1/N$ often leads the correct qualitative and often quantitative understanding of the physically relevant theories with low N. In non-Abelian gauge theories, the large-N limit corresponds to the planar limit [25].

In chapter 2 we provide tools to investigate asymptotic safety, namely the functional renormalisation group method. We review the computation of n-point functions and we go through the formalism of Wilsonian renormalisation and Wilson's interpretation of Kadanoff's block spin idea [26, 27]. We derive the exact renormalisation group equation, and we draw a road map by briefly discussing the approximation schemes, calculation of the β-functions that describe the running behaviour of the couplings, and we end the chapter with the brief discussion of the scaling exponents which are the universal quantities.

In chapter 3 we make progress in addressing the first question that we posed. We investigate the effects of beyond marginal couplings (higher order couplings with negative mass dimension) on the asymptotic safety of gauge-Yukawa theory which has recently been studied in [22]. In other words we would like to ensure that it stays a predictive and fundamental theory. We present our results as an expansion in terms of a small expansion parameter ϵ. We look into the Veneziano limit where we have a large-N_f and large N_c limit, N_f and N_c being flavour and colour degrees of freedom, respectively. However the ratio of N_f and N_c is finite in the Veneziano limit. We find that all the higher order operators that we include in the system carry a fixed point value with an irrelevant direction (meaning they are not independent parameters), which makes the theory asymptotically safe. We also find that our results can be derived analytically and the resulting potential can be resummed, bringing a logarithmic contribution. Therefore we get closed expressions for the coupling constant fixed points to leading order in ϵ. We also look into scaling exponents and we show that they are independent of the regulator choice, and hence universal, up to order $\mathcal{O}(\epsilon)$.

Although it is an active research area with more work to do, other aspects of the asymptotically safe four dimensional gauge-Yukawa theories have been explored, such as some of the beyond the standard model implications [28, 29], cosmology applications [30, 31], implications away from four dimensions [32], as well as model building directions and phenomenological implications [33].

In chapter 4 we address the second question. The quest is to understand the nature of gravity. We propose a model to investigate the asymptotic safety in a large number of dimensions. So, we turn our attention from a very general gauge theory such as in chapter 3, to a specific truncation of quantum gravity, namely Einstein-Hilbert truncation. Here, the value that is analogous to the group number N is the number of space-time dimensions D. This is because the local gauge invariance group for gravity is $GL(D, \mathbb{R})$ [34]. For gravity four dimensions is not special. One motivation to use extra dimensions for gravity is that in theories with compact extra dimensions the Planck scale becomes lower [35]. Here we take the already-well-established β-functions of quantum gravity [10, 11, 14]. These are given as a function of space-time dimensions D. We perform a $1/D$ expansion and present the fixed points in their leading orders in $1/D$. We show that the fixed points exist in large-D with a finite radius of convergence. We also show that scaling exponents are consistent with various other studies.

In chapter 5, we give an outlook and reflect on how we made progress in answering the questions that we stated here. We summarise our results in this last chapter and give concluding remarks.

Chapter 2

Functional renormalisation group

The main idea of the renormalisation group theory is to understand how the strength of the interactions change with different energy scales (or equivalently, length scales). For example if we look at small length scales, interactions of quarks matter a lot. Whereas if we zoom out to look at the nucleus of an atom, we only see the interactions of protons with neutrons and the interactions of individual quarks matter less. If we zoom out further and look at a visible/macroscopic substance, interactions of quarks will be negligible. At this scale we would be interested in the interactions between two different atoms. If we zoom out further to large scales, like a cosmologist, different interactions will matter. The way to distinguish between the interactions that matter in different energy scales is to use the coarse graining method of the renormalisation group which integrates (averages) out as we go along the physical scales.

In this chapter we motivate and introduce the functional renormalisation group method and derive the Wetterich equation. We then talk about how to calculate the β-functions and the universal critical exponents of a theory.

2.1 Recap: The n-point function

In this section we give a quick overview of the derivation of the effective action and the n-point correlation function. The n-point function, given as

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle \,, \tag{2.1}$$

tells us all the information that we want to know about how the particles interact with each other.

The n-point correlation function can be defined as the weighted average of the product of n fields over all possible field configurations, such as

$$\langle\varphi(x_1)\dots\varphi(x_n)\rangle := \frac{\int \mathcal{D}\varphi\,\varphi(x_1)\dots\varphi(x_n)\,e^{-S[\varphi]}}{\int \mathcal{D}\varphi e^{-S[\varphi]}}. \tag{2.2}$$

Here we focus on a simple scalar field theory in Euclidean space-time. Details of Euclidean Field Theory can be found in many field theory books such as [36, 37, 38]. We do not give all the details here, but it is worth mentioning that we obtain the generating functional in Euclidean space-time by applying a Wick rotation. This is a rotation in the complex time plane where we move from Minkowski space-time to Euclidean. By doing so, we will obtain the equation in (2.4) and the argument of the exponential will always be real negative when the source J is switched off. This means that instead of oscillating from one path to another in the path integral formulation, all the jagged paths are exponentially suppressed [38]. Minkowskian Green's functions may be derived from their Euclidean counterparts by analytical continuation provided that they satisfy some conditions. These conditions are given in [39].

Returning to the equation 2.2, note that we used the notation

$$\int \mathcal{D}\varphi = \prod_n \int \mathcal{D}\varphi(x_n). \tag{2.3}$$

The measure preserves the symmetry $\varphi \to \varphi^U$ where U is the symmetry transformation that satisfies the invariance of the action $S[\varphi] \to S[\varphi^U] = S[\varphi]$. Therefore with the space-time discretisation we have $\int_\Lambda \mathcal{D}\varphi \to \int_\Lambda \mathcal{D}\varphi^U = \int_\Lambda \mathcal{D}\varphi$.

The Euclidean Schwinger functional is defined as $W[J] = \ln Z[J]$ where the partition function $Z[J]$ is

$$Z[J] = \int \mathcal{D}\varphi \exp\{-S[\varphi] + J\cdot\varphi\}. \tag{2.4}$$

$J(x)$ is the external source, and "$\cdot$" implies integration such that $J\cdot\varphi = \int \mathrm{d}^D x J(x)\varphi(x)$. From this equation we can obtain an expression for the n-point function in terms of the Schwinger functional. Hence equation (2.2) becomes

$$\langle\varphi(x_1)\dots\varphi(x_n)\rangle = \frac{1}{Z[J]}\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\bigg|_{J=0}. \tag{2.5}$$

The effective action is the Legendre transformation of the Schwinger functional, W[J]. This both helps store information about the theory in an effective way and also ensures that the effective action is *convex*, or in other words: $\frac{\delta^2\Gamma}{\delta\phi^2} \geq 0$. Note that we denote the classical field by ϕ and the quantum field by φ. The effective action is defined as

$$\Gamma[\phi] = \sup_J \left(-W[J] + J \cdot \phi\right), \tag{2.6}$$

where "sup" means that the source J is chosen such that $\Gamma[\phi]$ is approaching its supremum. At the supremum, i.e. $J = J_{\text{sup}}$, we get $\frac{\delta}{\delta J}\left(-W[J] + J \cdot \phi\right) = 0$. Using this we can find the expectation value of the field as the following,

$$\phi \equiv \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J} = \langle \varphi \rangle_J \,. \tag{2.7}$$

So the expectation value of the quantum field with the fluctuations at the source J is equal to the classical field. We note that the quantum equation of motion can be found as

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = J(x)\,. \tag{2.8}$$

Once solved, this equation describes the dynamics of the theory. We can therefore go on to obtain the exponential of the effective action without any explicit dependence on the source term, J.

If we plug the vertex expansion, given by

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^D x_1 \ldots \mathrm{d}^D x_n \Gamma^{(n)}(x_1, \ldots, x_n) \phi(x_1) \ldots \phi(x_n), \tag{2.9}$$

into

$$e^{-\Gamma[\phi]} = \int_\Lambda \mathcal{D}\varphi \exp\left\{-S[\phi + \varphi] + \int \frac{\delta \Gamma[\phi]}{\delta \phi}\varphi\right\}, \tag{2.10}$$

we can solve this integro-differential equation in terms of $\Gamma[\phi]$ and extract the Schwinger-Dyson equations for the one particle irreducible proper vertices, $\Gamma^{(n)}$. Note that, in (2.10) we shift the field as $\varphi \to \varphi + \phi$ so that the integration is over the fluctuations of the field.

2.2 A heuristic picture of the coarse graining

Here we demonstrate how the coarse graining scheme works for the renormalisation group by looking into Wilson's take on Kadanoff's block spin modelling of the famous

Ising Model [27, 26]. Imagine we have a two dimensional, perfect square block made of atoms that only interact with their closest neighbours. Let us now zoom into this picture where we are looking into a specific $n \times n$ square. This $n \times n$ box is interacting with its neighbour. We describe the average strength of their interaction by the coupling constant g. The physics of this system is described by a Hamiltonian $H(k, g)$ where k describes the scale or distance between the neighbouring boxes. Within this box we can further zoom out and check a larger sized box. How these boxes interact with their neighbours is described by a new Hamiltonian $H(k', g')$.

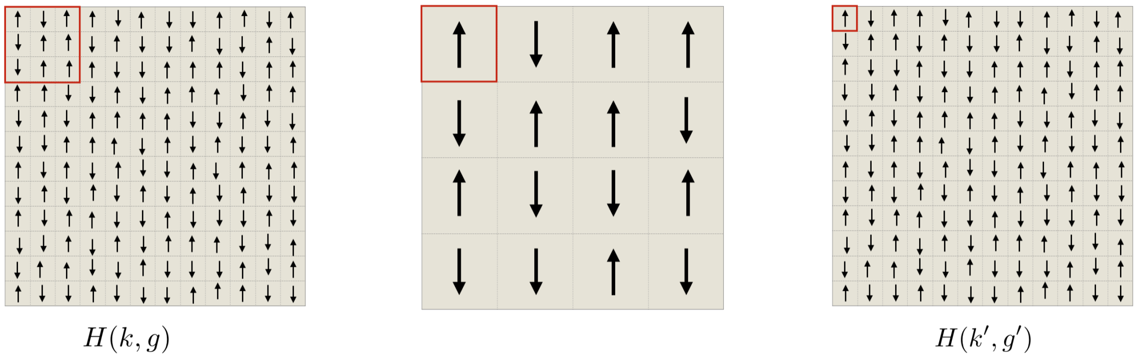

Figure 2.1: This image visualises how the averaging out and zooming out works in a block spin system in the Ising model. The red square in the second block is the averaged out version of the red square in the first block. And by zooming-out, in other words looking at the block at a different energy/length scale, we get the third block. The same red square can be seen there as well.

One can zoom into these boxes to look at the average interactions of smaller boxes, or zoom out to look at the average interactions of the bigger boxes, ends up with a series of Hamiltonians such as $H(k, g) \rightarrow H(k', g') \rightarrow H(k'', g'')$. These form a semi-group[1] since when the smaller ones are averaged out, they describe the same interaction. We give a visual demonstration of this in figure 2.1.

The simple yet sophisticated idea behind renormalisation group is the scale dependence of the theory. The interactions in the theory are described by the coupling constants which are scale dependent. Therefore the way the couplings depend on the scale determines the dynamics of the system in different energy scales.

[1]We call it semi-group and not a group, because the renormalisation group does not satisfy all the properties of a mathematical group. The scaling can be done only in one direction, once we integrated out, the information is lost. Therefore there is no inverse element.

It could be that the strength of the interactions become unreasonably large (i.e. tends to infinities, i.e. Landau pole). Alternatively, they may go to zero (meaning non-interacting, or asymptotically free). A third option is that they go to a fixed value. After this energy scale, whether we go to higher energy scales, the strength of the interaction does not change. This is called asymptotic safety. In terms of block-spin example, this would correspond to zooming out and getting the same spin distribution on the new block.

2.3 Effective average action and the Wetterich equation

To investigate the theories that appear in the later chapters of this thesis, we use the functional renormalisation group method. This non-perturbative method makes use of an functional renormalisation group equation refered to as the *Wetterich equation* or simply *the flow equation* [8]. In this section we will derive the Wetterich equation. We first start by discussing the properties of the effective average action. Here we follow some useful reviews such as [40, 41, 42] to accomplish this.

Instead of integrating out all the fluctuations at once, here we implement Wilson's idea of renormalisation, integrating out shell by shell the large momentum modes. We introduce the control parameter k, which is the energy scale. We attach this k, to all of our functions and variables to tell that they are "scale-dependent". For example the effective action becomes Γ_k, where in the extreme limit $k \to \infty$ we get the bare action: $\Gamma_{k\to\infty} = S_{\text{bare}}$. In the other extreme, when $k \to 0$, we have the full quantum effective action $\Gamma_{k=0} = \Gamma$. We call Γ_k *the Effective Average Action* (EAA).

These properties of the effective average action are regulated by a regulator function R_k. In general the regulating function is a mathematical trick to ensure that the EAA interpolates between the low and high momentum modes. Hence, we introduce an IR regulator piece ΔS_k into the total action.

$$Z_k[J] = \exp\left(-\Delta S_k \frac{\delta}{\delta J}\right) Z[J] \tag{2.11}$$

$$= \int_\Lambda \mathcal{D}\varphi \exp\left\{-S[\varphi] - \Delta S_k + J \cdot \varphi\right\} \tag{2.12}$$

This regulator functional has the form

$$\Delta S_k = \frac{1}{2}\int \frac{\mathrm{d}^D q}{(2\pi)^D}\varphi(-q)R_k(q)\varphi(q) \tag{2.13}$$

in Fourier space.

In order for the regulator to be able to do its job, it has to have the properties stated below:

- $\lim\limits_{q^2/k^2\to 0} R_k(q) > 0$. This ensures that the effective propagator remains finite, so that there are no infrared (IR) divergences even with the massless modes. This property shows that R_k is an IR regulator.

- $\lim\limits_{k^2/q^2\to 0} R_k(q) = 0$, so that the regulating part of the action is removed when $k \to 0$. In this limit, this property ensures that the EAA goes to quantum effective action (i.e. $\Gamma_k \to \Gamma$).

- $\lim\limits_{k^2\to\infty} R_k(q) \to \infty$, so that the quantum corrections are suppressed and we achieve the bare action as $k \to \infty$ (i.e. $\Gamma_{k\to\infty} \to S_{\text{bare}}$).

There is freedom in the choice of the regulator, as long as it satisfies the above conditions, since the physical theory does not depend on the choice of the regulator. In particular, it may be either a smooth regulator or a sharp one. An example for a smooth regulator can be $R_k(q) = q^2[\exp(q^2/k^2) - 1]^{-1}$ where the function goes from one region to another "smoothly". An example to a sharp regulator can be step function where the transition between two regions is "sharp".

Equivalent to the previous computation we get the following for the effective average action

$$\Gamma_k = \sup_J \{-W_k + J\cdot\phi\} - \Delta S_k[\phi]. \tag{2.14}$$

We should note here that even though the sup part of the expression is guaranteed to be convex, the ΔS_k part is not necessarily so.

From here we get a very similar equation to (2.10), for the effective average action,

$$e^{\Gamma_k[\varphi]} = \int \mathcal{D}\varphi \exp\left\{-S[\varphi+\phi] + \frac{\delta\Gamma_k}{\delta\phi}\varphi - \frac{1}{2}\varphi\cdot R_k\cdot\varphi\right\}. \tag{2.15}$$

The equivalent of the equation of motion in (2.8) can be similarly computed, now with the regulator term.

$$J(x) = \frac{\delta \Gamma_k[\phi]}{\delta \phi(x)} + R_k \phi(x), \tag{2.16}$$

$$\implies \frac{\delta J(x)}{\delta \phi(y)} = \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(y)\, \delta \phi(x)} + R_k. \tag{2.17}$$

Recalling that $\langle \varphi(x) \rangle = \phi(x) = \frac{\delta W[J]}{\delta J(x)}$, we find the two point function (i.e. the Green's function or the propagator) by taking the second functional derivative of the Schwinger functional with respect to the source term. In other words

$$\frac{\delta \phi(x)}{\delta J(x')} = \frac{\delta^2 W[J]}{\delta J(x') \delta J(x)} := G_k(x - x') = \langle\, \varphi(x)\, \varphi(x')\, \rangle - \langle \varphi(x) \rangle \langle \varphi(x') \rangle. \tag{2.18}$$

By looking at the equations (2.17) and (2.18), one being the inverse of the latter, we can see where this is going. Let us formally show that this is the case. Let us define the Dirac Delta function as the following.

$$\begin{aligned} \delta(x - x') = \frac{\delta J(x)}{\delta J(x')} &= \int \mathrm{d}^D y\, \frac{\delta J(x)}{\delta \phi(y)} \frac{\delta \phi(y)}{\delta J(x')} \\ &= \int \mathrm{d}^D y \left\{ \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(x) \delta \phi(y)} + R_k(x, y) \right\} G_k(y - x') \end{aligned} \tag{2.19}$$

We denote the second functional derivative of the EEA as $\Gamma_k^{(2)} := \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(x) \delta \phi(y)}$. And so we can deduce that

$$G_k(y - x') = \left(\Gamma_k^{(2)} + R_k \right)^{-1} (x, y) \tag{2.20}$$

Now recall the definition of the EAA such that $\Gamma_k = -W_k + J \cdot \varphi - \Delta S_k$. At $J = J_{\mathrm{sup}}$, we have

$$\partial_t \Gamma_k = -\partial_t W_k - \partial_t \Delta S_k. \tag{2.21}$$

Note that here (∂_t) is a short hand notation for $\left(k \frac{\partial}{\partial k}\right)$, since $t = \ln k$. Also note that throughout this thesis we will use this notation such that it only acts on the function immediate to its right, unless there are parentheses.

From the path integral formalism we can deduce the following.

$$e^{W_k} = \int \mathcal{D}\varphi e^{-S + \Delta S_k + J \cdot \varphi} \tag{2.22}$$

$$\implies \partial_t W_k e^{W_k} = \int \mathcal{D}\varphi\; \partial_t \left(-S + \Delta S_k + J \cdot \varphi \right)\; e^{-S + \Delta S_k + J \cdot \varphi} \tag{2.23}$$

$$\implies \partial_t W_k = -\frac{1}{2} \langle \varphi \cdot \partial_t R_k \cdot \varphi \rangle \tag{2.24}$$

Plugging (2.21) into (2.24), we get

$$\partial_t \Gamma_k = \frac{1}{2}\langle \varphi \cdot \partial_t R_k \cdot \varphi \rangle - \partial_t \Delta S_k \tag{2.25}$$

$$= \frac{1}{2}\langle \varphi(x) \cdot \partial_t R_k \cdot \varphi(y) \rangle - \frac{1}{2}\langle \varphi(x)\rangle \partial_t R_k \langle \varphi(y)\rangle \tag{2.26}$$

$$= \frac{1}{2}\partial_t R_k \big(\langle \varphi(x)\varphi(y)\rangle - \langle \varphi(x)\rangle\langle \varphi(y)\rangle\big) \tag{2.27}$$

$$= \frac{1}{2}\partial_t R_k G_k(y-x) \tag{2.28}$$

And now once again plug (2.20) into (2.28), we get *the Functional Renormalisation Group Equation* (FRGE). We shall also refer to it as *the Wetterich equation* or *the flow*.

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\partial_t R_k \left(\Gamma_k^{(2)} + R_k\right)^{-1}\right]. \tag{2.29}$$

Here the trace, Tr, means that the right hand side of the equation is integrated over the momentum and summed over all the eigenvalues of $\Gamma_k^{(2)}$ which is the second derivative of the effective average action with respect to the fields in the theory. Throughout the thesis, we occasionally refer to the right hand side (RHS) and the left hand side (LHS) of the FRGE as appears in equation (2.29). We showed this derivation for the simplest case, a real scalar field. Generalisations of this to other gauge theories can be made.

The Wetterich equation is very important in the sense that it is an exact equation which has the form of a one-loop propagator but contains information about all loop orders. Up to this point, to derive the FRGE we have not made any approximations (hence, it is exact). Having said that, it is not possible to analytically solve this equation without introducing approximation schemes and truncate. The approximation schemes can vary from vertex expansion, to perturbative expansion, or to derivative expansion [43, 44, 45]. On top of this, we can use a large-N limit [46, 25], or use the space-time dimensions as the expansion parameter [47]. We will explain what approximations we use to compute the relevant flow in the relevant chapters.

2.4 β-functions and fixed points

The next two sections serve as a sneak-peak of what is coming in the following chapters of this thesis. Here we give a brief explanation of the β-function and the fixed points, hoping that the reader will appreciate what is coming in the following chapters where we explain the details more thoroughly for the specific theories.

Throughout this thesis, we take the regulating function R_k as the optimised cutoff (also referred to as the Litim cutoff) [13, 12] , which enables us to compute the flow analytically for the truncations that the theories in question include. This is given by

$$R_k(q) = (k^2 - q^2)\Theta(k^2 - q^2), \tag{2.30}$$

which can easily be shown to satisfy the conditions for the regulator, given in the previous section.

The information about the running behaviour of the coupling constants is contained in a set of differential equations called the β-functions. For a given action

$$\Gamma_k[\phi] = \sum_i g_i\, \mathcal{O}_i, \tag{2.31}$$

the g_is represent the coupling constants which describe the interactions. The $\mathcal{O}_i$s represent the associated operators. We would like to compute the β-functions for each of the coupling constants, i.e. $\beta_i = \partial_t g_i$. We might find fixed points if we can solve the equations $\beta_i = 0$, which tells us where the running stops.

β-functions can be computed from the FRGE in equation (2.29). We can compute the flow of the potential, i.e. $\partial_t v_k$ (where v_k is the dimensionless, k-dependent potential), from the Wetterich equation, for a general form of the action. This flow contains the information about the β-functions.

In this thesis, we compute the β-functions by introducing an ansatz for the action Γ_k (and the potential V_k). This ansatz for the potential is usually in the form of a series expansion in powers of the field with associated coupling constants of the interactions as the coefficients of this expansion.

Let us remind ourselves that we said in previous sections that one of the most important properties of the functional renormalisation method is that we write all the quantities in a dimensionless (or equivalently, scale-dependent) form. Now we give a very simple example to demonstrate how the β-functions can be computed from the FRGE equation (2.29) by using dimensionless quantities and how their scaling comes into the calculation. For instance

$$V_k = k^d v_k, \tag{2.32}$$

where V_k is a generic potential which has canonical mass dimensions, d. So, the lowercase v_k is the dimensionless potential.

On the LHS of the FRGE we have the scale derivative of the EAA. Therefore we need to take into account the canonical scaling of the potential when we are computing the flow.

$$\partial_t V_k = \partial_t (k^d v_k). \tag{2.33}$$

Hence, on the LHS of the FRGE, we will have the scale derivative of the potential, which will be an expansion in powers of the field with the β-functions of the related couplings as the coefficients of the fields. We will have terms coming from the rescaling of the potential. Once rescaled, we will have the general form

$$\partial_t v_k = -d\, v_k + \text{RHS of FRGE}, \tag{2.34}$$

where "RHS of FRGE" is in its integral form[2]. So we can say that the terms that are coming from the canonical scaling of the potential contributes to the classical scaling. On the other hand the terms that are coming from the RHS of the FRGE contribute to the quantum scaling.

In order to find the nth β-function, β_n, we take the nth derivative of the right hand side of equation (2.34) with respect to all the fields that are included in the interaction that the associated coupling describes, i.e. all the fields that are multiplied with the coupling constant. We then set the fields to zero to get an expression for the β-function in terms of the coupling constants. To describe what we said in words in mathematical terms:

$$\partial_{\phi_i} \ldots \partial_{\phi_n} \left(\partial_t v_k\right)\big|_{\phi_j=0} = \beta_n, \tag{2.35}$$

where ϕ_i to ϕ_n are the fields that are dressed with the nth coupling constant in the potential ansatz and ϕ_j represents all the fields.

[2]Once the integration is performed, the volume element and loop factors will come out of the integral. Here we give a general form. We need to know about the specific action to be able to compute the functional integrals. We will give more details about the computation for specific theories in the coming chapters.

A simple example: Let g_n define the coupling for a term in the potential such as $g_n\phi_2\phi_5\phi_6$. Therefore in order to find the β-function of the coupling g_n, we need to take the derivative of this term with respect to ϕ_2, ϕ_5, ϕ_6 and k. So

$$\partial_{\phi_2}\partial_{\phi_5}\partial_{\phi_6}\left(\partial_t(g_n\phi_2\phi_5\phi_6)\right)\Big|_{\phi_j=0} = \partial_t g_n = \beta_n. \tag{2.36}$$

Asymptotic safety is investigated by finding the fixed points of the theory, i.e. by solving the equations $\beta_n = 0$. If we find fixed points, it means the theory is going to a finite, well-defined place in the UV, therefore it is safe.

2.5 Critical exponents

The fixed point values themselves are not universal. The values they take might change with a different field redefinition or if we were to take different assumptions. What is universal are the critical exponents. The critical exponents tell us how the flow behaves around a given fixed point. This behaviour carries a lot of importance in terms of the predictivity of the theory. This is investigated by calculating the scaling exponents of the system.

We take the linearised flow around the fixed point. This can be defined as

$$\partial_t\, g_i = \beta_i \approx \sum_j \frac{\partial \beta_i}{\partial g_j}\bigg|_{g_j^*} (g_j - g_j^*). \tag{2.37}$$

We name the term, $\frac{\partial\beta_i}{\partial g_j}\Big|_{g_j^*} := M_{ij}$, as the *stability matrix*. We name the eigenvectors of this matrix as W_i and the eigenvalues as $-\vartheta_n$. Therefore we can find the solution to equation (2.37) as

$$g_i = g_i^* + \sum_n C_n W_i^n \left(\frac{k}{k_0}\right)^{-\vartheta_n}, \tag{2.38}$$

where C_n are the constants of integration and k_0 is a reference scale. Either of the three conclusions can be drawn depending on the sign of the eigenvalues.

- If the eigenvalue, $(-\vartheta_n)$, has negative real part, we have a *relevant* operator or a relevant direction.

- If the eigenvalue, $(-\vartheta_n)$, has positive real part, we have an *irrelevant* operator or an irrelevant direction.

- If the real part of $(-\vartheta_n)$ vanishes, we have a *marginal* operator. In this case we need to go beyond the linear approximation in equation (2.37).

The number of relevant operators are the number of observables that we need to be given to make a prediction. For a given theory to be predictive we would like to have a finite amount of relevant operators. The flows of the relevant operators come out from an IR fixed point and go to a UV fixed point. In other words they are "IR repulsive" or "UV attractive". Therefore to start with we need to have the knowledge of this coupling constant from a measurement in the IR. Whereas the irrelevant directions flow from the UV to IR fixed point. Therefore we can use the relevant operator to predict the irrelevant coupling values. This is why we need to have a finite number of relevant directions for our theory to be predictive.

The surface that is spanned by the irrelevant directions is called the critical surface. In order to have a predictive theory, we need to have a finite dimensional critical surface, so that we have a finite number of free parameters to fix in a given theory. A slight shift away from the critical surface would put us on a flow line that is a repulsive direction.

Here, in relation to the directions of the flow, a more detailed explanation of the integration constant in (2.37) is necessary. In order to be at the fixed point, C_n needs to be zero. So when $C_n = 0$ and we have an irrelevant direction, we are on the critical surface. When we have a relevant direction, we are attracted towards the UV fixed point, independent of the value of C_n. Therefore in this case the C_n's are the free parameters that we need to fix.

Chapter 3

Gauge theories with beyond marginal operators in the large-N limit

Although asymptotic safety has emerged as a solution to the non-renormalisability of quantum gravity, as a measure of predictibility, asymptotic safety of types of theories other than quantum gravity has also gained a lot of interest. The reason being, first of all, theory having fixed points is a good predicament about its predictibility and shows that the theory is a fundamental theory of physics. Secondly, establishing a good, stable picture with theories other than quatum gravity will improve the reliability of asymptotic safety on quantum gravity. A sample of the literature on asymptotic safety of other non-perturbatively renormalisable theories (other than quantum gravity) are [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 28, 58, 59, 60].

More recent studies have shown that asymptotic safety can be achieved within perturbative approaches to four dimensional gauge and Yukawa theories [22, 23]. In this chapter we investigate whether the recently discovered UV fixed points in four-dimensional gauge theories coupled to matter persists beyond the level of classically marginal couplings. The challenge with this is that one has to deal with infinitely many new invariants. Moreover, higher order invariants have coupling constants with increasingly negative mass dimensions.

We investigate the gauge-Yukawa system introduced in [22] specifically with consideration of the contribution of higher order scalar self-interactions. Since these higher derivative operators have negative mass dimensions we cannot renormalise them by using the famously successful perturbation theory. Hence we are in the realm of non-perturbative renormalisation and the functional renormalisation group which we introduced in the previous chapter.

We also do our investigation in the so-called *Veneziano Limit*, first introduced in [61] as an alternative to 't Hooft's large-N_c formalism, in order to make the chiral $U(1)$ anomaly visible. Different from 't Hooft's large-N_c, in Veneziano limit both N_f and N_c (number of flavour degrees of freedom and number of colour degrees of freedom respectively) are taken large. But the ratio N_f/N_c is finite. Increasing the degrees of freedom simplifies the system by enabling us to do an expansion around a small variable ϵ defined as $\epsilon = N_f/N_c - 11/2$ where $0 < \epsilon \ll 1$. This limit is interesting because at $\epsilon > 0$, asymptotic freedom is lost. A more detailed discussion of the features of this limit can be found in [62].

We give a quick recap of the results of reference [22]. Then we go on to show the derivations of the flow equations, using the methods introduced in the previous chapter. We then discuss our numerical and analytical results for the remainder of this chapter.

3.1 Recap: Asymptotically safe gauge theories

We begin by briefly recalling the basics of the asymptotically safe gauge-Yukawa model in four dimensions investigated in [22]. This theory contains $SU(N_c)$ gauge fields[1] A^a_μ with field strength $F^a_{\mu\nu}$ ($a = 1, \cdots, N_c$), N_f flavors of fermions Q_i ($i = 1, \cdots, N_f$) in the fundamental representation, and an $N_f \times N_f$ complex scalar matrix H which is uncharged under the gauge group. In the massless limit the action consists of the Yang-Mills action, the fermion kinetic term, the Yukawa coupling,

[1]Note that the number of group generators is $(N_c^2 - 1)$. In the standard model, this is analogous to the eight gluons of $SU(3)$ QCD.

and the scalar kinetic and self-interaction $L = L_{\rm YM} + L_{\rm kin.} + L_{\rm Yuk.} + L_{\rm pot.}$, where

$$\begin{aligned} L_{\rm YM} &= -\tfrac{1}{2}{\rm Tr}\, F^{\mu\nu} F_{\mu\nu} \\ L_{\rm kin.} &= {\rm Tr}\left(\overline{Q}\, i\not{D}\, Q\right) + {\rm Tr}\left(\partial_\mu H^\dagger\, \partial^\mu H\right) \\ L_{\rm Yuk.} &= -y\, {\rm Tr}\left(\overline{Q}\, H\, Q\right) \\ L_{\rm pot.} &= -u\, {\rm Tr}\left(H^\dagger H\, H^\dagger H\right) - v\left({\rm Tr}\, H^\dagger H\right)^2 . \end{aligned} \tag{3.1}$$

The Tr denotes the trace over both color and flavor indices. The model has four classically marginal coupling constants given by the gauge coupling g, the Yukawa coupling y, and the quartic scalar couplings u and the "double-trace" scalar coupling v, which we write as

$$\alpha_g = \frac{g^2\, N_c}{(4\pi)^2}\,, \quad \alpha_y = \frac{y^2\, N_c}{(4\pi)^2}\,, \quad \alpha_h = \frac{u\, N_f}{(4\pi)^2}\,, \quad \alpha_v = \frac{v\, N_F^2}{(4\pi)^2}\,. \tag{3.2}$$

In order to achieve exact perturbative control we take the Veneziano limit by sending both N_c and N_f to infinity while keeping their ratio N_f/N_c constant. The latter becomes a freely tunable parameter and we are particularly interested in the regime where

$$0 \leq \epsilon \equiv \frac{N_f}{N_c} - \frac{11}{2} \ll 1, \tag{3.3}$$

which is a prerequisition for an asymptotically safe fixed point within the perturbative regime. In terms of (3.3), and to leading non-trivial order in perturbation theory, the running of couplings takes the form [22]

$$\begin{aligned} \beta_g &= \tfrac{4}{3}\epsilon\, \alpha_g^2 + \left(25 + \tfrac{26}{3}\epsilon\right) \alpha_g^3 - 2\left(\tfrac{11}{2} + \epsilon\right)^2 \alpha_g^2\, \alpha_y \\ \beta_y &= (13 + 2\epsilon)\, \alpha_y^2 - 6\, \alpha_y\, \alpha_g \\ \beta_h &= -(11 + 2\epsilon)\, \alpha_y^2 + 4\alpha_h(\alpha_y + 2\alpha_h)\,, \\ \beta_v &= 12\alpha_h^2 + 4\alpha_v\left(\alpha_v + 4\alpha_h + \alpha_y\right)\,. \end{aligned} \tag{3.4}$$

Note that although the equations are given in terms of the expansion parameter ϵ, we have not assumed the limit yet.

We observe that the Yukawa coupling mixes with the scalar coupling when we include the two-loop Yukawa beta function contribution, and three-loop gauge beta function contribution. Also the gauge fields mix with scalar fields indirectly via Yukawa interactions at three-loop level. At this next-to-next-to-leading order (NNLO) approximation level in perturbation theory, we find the fixed point. To leading order in ϵ defined in equation (3.3), the beta functions in equation (3.4) display an interacting fixed point of order ϵ in all classically marginal couplings[2],

$$\alpha_g^* = \tfrac{26\epsilon}{57}\,, \quad \alpha_y^* = \tfrac{4\epsilon}{19}\,, \quad \alpha_h^* = \tfrac{\epsilon}{19}(\sqrt{23}-1)\,, \quad \alpha_v^* = -\tfrac{\epsilon}{19}\left(2\sqrt{23}-\sqrt{20+6\sqrt{23}}\right)\,, \tag{3.5}$$

where the asterix symbol, $*$ represents a fixed point.

We notice that the parametric smallness of the gauge coupling entails the smallness of the Yukawa and the quartic couplings. Moreover, vacuum stability of the theory is confirmed owing to [22, 63]

$$\alpha_h^* + \alpha_v^* = \tfrac{\epsilon}{19}(\sqrt{20+6\sqrt{23}}-\sqrt{23}-1) > 0\,. \tag{3.6}$$

When we look at the eigenvalues of the stability matrix of these interactions, we see that the eigenvalue dominated by the gauge interactions is the only relevant eigenvalue (i.e. $\vartheta < 0$) of this theory.

$$\vartheta_g = -\tfrac{104}{171}\,\epsilon^2\,, \quad \vartheta_y = \tfrac{52}{19}\,\epsilon\,, \quad \vartheta_1 = \tfrac{16}{19}\sqrt{23}\,\epsilon\,, \quad \vartheta_2 = \tfrac{8}{19}\sqrt{20+6\sqrt{23}}\,\epsilon\,. \tag{3.7}$$

Consequently, the UV critical surface is one dimensional, and all other couplings are determined by the gauge coupling along the UV safe trajectories emanating out of the fixed point.

This work carries a lot of importance in terms of showing that four-dimensional gauge-Yukawa theories interacting with scalars have UV-fixed points and are asymptotically safe. In our work we would like to investigate the effect of higher dimensional operators on this theory.

With regards to this theory, the questions we would like to answer in the coming sections are

- Does the higher dimensional scalar couplings change the fixed point structure of this theory?

- Can we still attain asymptotic safety with the higher dimensional operators included?

[2] Here we give the numerical values of these fixed point values. $\alpha_g^* = 0.45614\epsilon$, $\alpha_y^* = 0.210526\epsilon$, $\alpha_h^* = 0.199781\epsilon$, and $\alpha_v^* = -0.13725\epsilon$.

3.2 Beyond marginal operators

In this section we summarise the model that we will investigate in this chapter as well as introducing our notation convention.

We are interested in investigating the impact of higher dimensional operators on the scalar sector. Various invariants can be formed out of complex scalar $N_f \times N_f$ matrices. Besides the invariants $(\mathrm{Tr}\, H^\dagger H)^2$ and $\mathrm{Tr}\left(H^\dagger H\right)^2$ which already appear in the scalar potential (3.1) we may construct further invariants, as well as powers and cross-products thereof. We are specifically interested in higher order invariants of the form

$$V_k(P_1, P_2) = U_k(P_1) + P_2\, C_k(P_1), \tag{3.8}$$

where

$$U_k(P_1) = \sum_{n=1}^{\infty} A_{2n}\, P_1^{n+1}, \tag{3.9}$$

$$C_k(P_1) = \sum_{n=1}^{\infty} A_{2n-1}\, P_1^{n-1}, \tag{3.10}$$

and A_i represents the dimensionful coupling constants. The fields are defined as

$$P_1 = \left(\mathrm{Tr}\, H^\dagger H\right), \tag{3.11}$$

$$P_2 = \mathrm{Tr}\left(H^\dagger H - \frac{\mathbb{1}}{N_f}\mathrm{Tr}\, H^\dagger H\right)^2. \tag{3.12}$$

We can in fact include higher order combinations of P_1 and P_2 in equation (3.8). The reason we stop at this truncation level is simplicity.

We emphasised in the previous chapter that one of the most prominent properties of the renormalisation group is that the (classical and quantum) scaling of the quantities govern the dynamics of the theories. In order to make use of this we write everything in dimensionless (or in other words k-dependent) forms. We also introduce some redefinitions in order to perform the large-N_f expansion and to rid our equations of the π-dependence that comes from the loop calculations (purely for simplification purposes). Note that we denote dimensionful quantities in capital letters and the dimensionless quantities in lower case letters.

$$\alpha_{2n-2} = \frac{(4\pi)^{2n-2}}{N_f^{2n-2}}\, A_{2n-2} Z_H^{-n} k^{2n-4}, \tag{3.13}$$

$$\alpha_{2n+1} = \frac{(4\pi)^{2n+2}}{N_f^{2n+1}}\, A_{2n+1} Z_H^{-(n+2)} k^{n}, \tag{3.14}$$

where these couplings translate to the couplings defined in the previous section and in [22] as $\alpha_v = (\alpha_2 - \alpha_1)$ and $\alpha_h = \alpha_1$. So, we have two types of interactions described by the potentials U_k and C_k, where U_k describes P_1^n type of interactions, and c_k describes $P_2 P_1^n$ type of interactions. We make a distinction between the odd numbered couplings and the even numbered couplings. Namely, the part of the potential that is U_k contains the even-numbered couplings whereas, the part that is C_k contains the odd-numbered couplings.

Here we write the wave function renormalisation factors for completeness' sake. In case some curious person would like to investigate this theory with the anomalous dimensions included, they can use these equations as a reference. In this work we do not go beyond $Z_H = 1$ and $Z_Q = 1$.

We rescale the field and the potential as

$$P_1(H^\dagger, H) = N_f^2 \rho_1(h^\dagger, h) k^2, \tag{3.15}$$

$$U_k = k^4 u_k N_f^2, \tag{3.16}$$

$$C_k = \frac{c_k}{N_f}. \tag{3.17}$$

Note that we will be mostly using the dimensionless quantities in the following sections.

An important feature to note here is that we omit the mass term for simplicity. We are interested in the leading order behaviour in the expansion parameter ϵ and the mass term only brings about contribution to higher order terms in ϵ.[3]

3.3 The flow

Here we give a detailed computation of the flow for our specific theory as promised in the previous chapter.

3.3.1 The scalar flow

As we said, we are in the realm of the functional renormalisation group method, therefore we need the flow equations. In this section we give and explain the flow

[3]Note that the mass terms contribute to the denominators of the β-functions of higher order couplings as $(1 + m^2)$ and powers thereof. This only contributes to sub-leading terms in ϵ since the fixed point of the mass squared term has leading order ϵ itself.

equations as well as computing the β-functions of the scalar part of the gauge-Yukawa model that we introduced in the previous section. To do this we must make use of the FRGE that is given in equation (2.29).

The left hand side (LHS) of the FRGE is the scale derivative of the effective average action (EAA). In this case the scale derivative of the EAA is equivalent to the scale derivative of the dimensionful potential, $\partial_t V_k$. Given the dimensionless potential $v_k(\rho_1, \rho_2) = u_k(\rho_1) + \rho_2\, c_k(\rho_1)$, the LHS of the FRGE can be written independent of any Ansatz as

$$\text{LHS} = \partial_t u_k + 4u_k - 2\rho_1 u_k' + \rho_2 \left(\partial_t c_k - 2\rho_1 c_k'\right). \tag{3.18}$$

Note that ρ_1 and ρ_2 are the dimensionless redefinitions of the fields defined in equation (3.11) and (3.12).

We shall also need the RHS of the flow in order to solve the FRGE. We start by parameterising the dimensionless complex scalar field h, which is an $N_f \times N_f$ matrix.

$$h = \{h_{\mathrm{ab}}\}, \tag{3.19}$$

$$h_{\mathrm{ab}} = h_{\mathrm{a}} \delta_{\mathrm{ab}}, \tag{3.20}$$

where the indices represent each element of the scalar field and δ_{ab} is the Kronecker-delta. Here we make the assumption that h_{ab} is a diagonalisable matrix.

The Hessian in the denominator of the FRGE in (2.29), means the following.

$$\Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k}{\delta H_{\mathrm{ab}}^{\mathrm{I/R}} \delta H_{\mathrm{cd}}^{\mathrm{I/R}}} = q^2 + \frac{\delta^2 V_k}{\delta H_{\mathrm{ab}}^{\mathrm{I/R}} \delta H_{\mathrm{cd}}^{\mathrm{I/R}}}. \tag{3.21}$$

where q^2 is the four-momentum squared. $H_{\mathrm{ab}}^{\mathrm{I/R}}$ represents the imaginary (H^{I}) or the real (H^{R}) components of the (dimensionful) scalar field H. The mass spectrum, in other words the eigenstates of equation (3.21) need to be computed.

We can seperate the flow into two pieces, namely $\partial_t u_k$ and $\partial_t c_k$. Since the c_k part of the potential is linear in ρ_2, and this is the only ρ_2-dependence of the potential, a separation is possible. In the main body of this thesis we focus on the computation of the u_k part of the flow, $\partial_t u_k$, by setting $\rho_2 = 0$, because this is a simpler calculation for us to follow here. We give the raw material for a $\partial_t c_k$ recipe in the appendix A and we will comment on this later in this section.

The mass eigenstates of U_k can be computed as [64]

$$M_0^2 = k^2 + U_k' \,, \tag{3.22}$$
$$M_1^2 = k^2 + U_k' + \frac{4P_1}{N_f} C_k \,, \tag{3.23}$$
$$M_2^2 = k^2 + 2P_1 U_k'' + U_k'. \tag{3.24}$$

with multiplicities $(N_f^2 - 1)$, 1, and N_f^2 respectively. Details of this part of the scalar mass spectrum computation can be found in [64]. Here we see that even though we ignore the ρ_2(or P_2)-type interactions, c_k does not completely distangle itself from the set of equations, it appears in M_1^2.

Having the mass spectrum we can now solve the functional integral such as

$$\mathrm{RHS} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}^D q}{(2\pi)^d}\left\{ \frac{\partial_k R_k(q)\left(N_f^2 - 1\right)}{P_k(q) + U_k'} + \frac{\partial_k R_k(q)}{P_k(q) + U_k' + \frac{4P_1}{N_f} C_k} + \frac{\partial_k R_k(q) N_f^2}{P_k(q) + U_k' + 2P_1 U_k''} \right\} \tag{3.25}$$

where $P_k = q^2 + R_k(q)$ and we remind the reader that, in this thesis, we always take $R_k(q) = (k^2 - q^2)\Theta(k^2 - q^2)$.

Next, to simplify the integration, we define a new integration variable as $y = \frac{q^2}{k^2}$. Hence $R_k(q) = y\, k^2 r(y)$ where

$$r(y) = \left(\frac{1}{y} - 1\right)\Theta\left(1 - y\right) \tag{3.26}$$

and

$$\partial_t R_k(q) = \partial_t \left(y k^2 r(y)\right) = -2y^2 r'(y) k^2. \tag{3.27}$$

We also write the potential in the dimensionless parameterisation as in equations (3.15) to (3.17).

Finally, the integration measure changes first as $\int \mathrm{d}^D q = \int \mathrm{d}\Omega_4 \int \mathrm{d}q q^3$ by Wick rotation, where $\int \mathrm{d}\Omega_4 = v_4 = 2\pi^2$ is the four dimensional volume element. And secondly, with the change of variable, q to y, the integral becomes $\int \mathrm{d}q q^3 = \frac{k^4}{2}\int \mathrm{d}y y$. Overall, by solving equation (3.25) with the change of variables, we get the RHS of the flow equation for the u_k part of the potential.

The computation of $\partial_t c_k$ has a very similar method, only the details are much more complicated that it is not in the scope of this thesis, although these details can be found in references such as [64] and [65]. We can use the mass spectrum given in the appendix A to compute the full flow $\partial_t v_k = \partial_t u_k + \rho_2 \partial_t c_k$ in a similar manner to the flow of only $\partial_t u_k$, by solving the functional integral. Since c_k is linear in ρ_2, we can safely say that the terms multiplied with ρ_2 on the RHS are a part of the flow of c_k and the rest are a part of the flow of u_k.

We are interested in the Veneziano limit where we can reach asymptotic safety in four dimensional gauge-Yukawa theories as we mentioned in the earlier section. The Veneziano limit is where we have a large number of flavour and colour degrees of freedom, N_f and N_c respectively, but the ratio of the two is finite. After simplifying the flow in the large-N_f limit it is found [64]

$$\mathrm{RHS}_1 = \frac{1}{2}\left(\frac{1}{1+u_k'+4\rho_1 c_k} + \frac{1}{1+u_k'}\right), \tag{3.28}$$

$$\begin{aligned}\mathrm{RHS}_2 = \frac{1}{2}\Bigg(&\frac{64\rho_1^2 c_k^3\left(c_k-\rho_1 c_k'\right)}{\left(1+u_k'\right)^2\left(1+u_k'+4\rho_1 c_k\right)^3} - \frac{48\rho_1^2 c_k^2 c_k'}{\left(1+u_k'\right)\left(1+u_k'+4\rho_1 c_k\right)^3}\\ &-\frac{8\rho_1 c_k c_k'}{\left(1+u_k'+4\rho_1 c_k\right)^3} - \frac{2c_k'}{\left(1+u_k'+4\rho_1 c_k\right)^2}\\ &+\frac{128\rho_1^3 c_k^5}{\left(1+u_k'\right)^3\left(1+u_k'+4\rho_1 c_k\right)^3} + \frac{16c_k^2}{\left(1+u_k'+4\rho_1 c_k\right)^3}\Bigg).\end{aligned} \tag{3.29}$$

where $\mathrm{RHS}_1 + \rho_2 \mathrm{RHS}_2$ gives the whole RHS of the FRGE. But this is not the end of story. We still need to include the Yukawa contribution to the flow. Next, we explain how to take the Yukawa contribution into account.

3.3.2 The Yukawa contribution

The contribution of the Yukawa sector to the scalar flow can be extracted from previous perturbation theory results [22]. Let us remind the reader how we defined the β-function in the previous chapter, see equation (2.35). In order to find the same β-functions given in (3.4), we need to add the Yukawa contribution into the RHS of FRGE such that the contribution to β_1 is multiplied with ρ_2 and contribution to β_2 is multiplied with ρ_1^2. This way we ensure that, when the relevant derivatives are taken as explained in the previous chapte equation (2.36), the Yukawa contribution is accounted for in the β-functions of the theory.

Therefore the gauge-Yukawa contribution to the functional flow of the scalar sector can be written as

$$\Delta_{\mathrm{Yuk.}} = \rho_1^2\left(4\alpha_y\alpha_2 - \alpha_y^2(11+2\epsilon)\right) + \rho_2\left(4\alpha_y\alpha_1 - \alpha_y^2(11+2\epsilon)\right) \tag{3.30}$$

Then combining equations (3.18), (3.28), (3.29), and (3.30) we calculate the flow of u_k and c_k as

$$\partial_t u_k = \underbrace{-4u_k+2\rho_1 u_k'}_{\text{Rescaling}} + \underbrace{\left(4\alpha_2\alpha_y - (11+2\epsilon)\alpha_y^2\right)\rho_1^2}_{\text{Yukawa contribution}} + \underbrace{\frac{1}{2}\left(\frac{1}{1+u_k'} + \frac{1}{1+u_k'+4\rho_1 c_k}\right)}_{\text{RHS of the FRGE}} \tag{3.31}$$

$$\begin{aligned}\partial_t c_k = &\underbrace{2\rho_1 c_k'}_{\text{Rescaling}} + \underbrace{\left(4\alpha_y\alpha_1 - \alpha_y^2(11+2\epsilon)\right)}_{\text{Yukawa contribution}} \\ &\underbrace{\begin{aligned}&+\frac{1}{2}\left(\frac{64\rho_1^2 c_k^3\left(c_k-\rho_1 c_k'\right)}{\left(1+u_k'\right)^2\left(1+u_k'+4\rho_1 c_k\right)^3} - \frac{48\rho_1^2 c_k^2 c_k'}{\left(1+u_k'\right)\left(1+u_k'+4\rho_1 c_k\right)^3}\right. \\ &-\frac{8\rho_1 c_k c_k'}{\left(1+u_k'+4\rho_1 c_k\right)^3} - \frac{2c_k'}{\left(1+u_k'+4\rho_1 c_k\right)^2} \\ &\left.+\frac{128\rho_1^3 c_k^5}{\left(1+u_k'\right)^3\left(1+u_k'+4\rho_1 c_k\right)^3} + \frac{16c_k^2}{\left(1+u_k'+4\rho_1 c_k\right)^3}\right)\end{aligned}}_{\text{RHS of the FRGE}}\end{aligned} \tag{3.32}$$

These equations have a central place in our study. We will use them to compute the β-functions and the fixed points in the coming sections of this chapter. A little bit of information is in place to explain out underlining. The *"Rescaling"* terms are the terms coming from canonical rescaling of the potential. The *"Yukawa contribution"* is as explained above, extracted from the perturbation results and separated so that they have the right contribution to the β-functions. Finally the *"RHS of FRGE"* terms are the quantum part of our theory, terms computed using the FRGE.

3.4 Numerical results

For the Ansatz given in equations (3.9) and (3.10), we use the flow equations in (3.31) and (3.32) to compute the β-functions. The LHS of equation (3.31) and (3.32) are an expansion in terms of the field ρ_1 and the β-functions are their coefficients. Similarly, on the RHS, we should match the expressions that are multiplied with the same fields with the β-functions. For this we use equation (2.35) that we introduced in the previous chapter. We find the β-functions as

$$\beta_1 = \frac{16\alpha_1\epsilon}{19} + 8\alpha_1^2 - \alpha_3 - \frac{16}{361}\epsilon^2(2\epsilon+11) \tag{3.33}$$

$$\beta_2 = -\frac{16}{361}\epsilon^2(2\epsilon+11) - \frac{16\alpha_2\epsilon}{19} + 4\left(2\alpha_1^2 + 2\alpha_2\alpha_1 + \alpha_2^2\right) - 2\alpha_3 - 3\alpha_4 \tag{3.34}$$

$$\beta_3 = 2\alpha_3 - 24(4\alpha_1 + 2\alpha_2)\alpha_1^2 + 12\alpha_3\alpha_1 + 2(4\alpha_1+2\alpha_2)\alpha_3 - 2\alpha_5 \tag{3.35}$$

$$\beta_4 = 2\alpha_4 - 4\alpha_2^3 + 6\alpha_4\alpha_2 - 2\alpha_1 - \alpha_2)^3 + 2\alpha_1 + \alpha_2)(8\alpha_3 + 6\alpha_4) - 2\alpha_5 - 4\alpha_6 \tag{3.36}$$

$$\begin{aligned}\beta_5 = {} & 4\alpha_5 + 800\alpha_1^4 + 768\alpha_2\alpha_1^3 + 24\left(8\alpha_2^2 - 13\alpha_3 - 3\alpha_4\right)\alpha_1^2 + 24(\alpha_5 - 5\alpha_2\alpha_3)\alpha_1 \\ & -12\alpha_2^2\alpha_3 + 6\alpha_3(2\alpha_3+\alpha_4) + 8\alpha_2\alpha_5 - 3\alpha_7\end{aligned} \tag{3.37}$$

$$\begin{aligned}\beta_6 = {} & 4\alpha_6 + 8\alpha_2^4 - 18\alpha_4\alpha_2^2 + 8\alpha_6\alpha_2 + 8(2\alpha_1+\alpha_2)^4 36\alpha_4^2 + \frac{1}{8}(8\alpha_3+6\alpha_4)^2 \\ & -6(2\alpha_1+\alpha_2)^2(4\alpha_3+3\alpha_4) + 8(2\alpha_1+\alpha_2)(\alpha_5+\alpha_6) - 2\alpha_7 - 5\alpha_8\end{aligned} \tag{3.38}$$

by identifying the LHS and the RHS of the flow equation as we described. Recall that equations (3.33) and (3.34) can be compared to the perturbative equations β_h and β_v, found in (3.4). We find even higher order β-functions than those given above, but it is not necessary to write them all here. Although we give the equations up to β_6 as an example here, the reader must note that we in fact extract these equations up to order β_{36}. The only reason we did not go beyond this is the computing time.

We also note that the fixed point of the Yukawa coupling is given as $\alpha_y^* = 4/19\epsilon$ to the leading order in ϵ. We can see this contribution in equations (3.33) and (3.34) where $16/19\epsilon$ can be identified as $4\alpha_y$ and $16/361\epsilon^2$ can be identified as α_y^2.

Next we solve the tower of β-functions of the higher order couplings, some of which are given in equations (3.33-3.38), to find the fixed points. We note that each β-function is a function of the couplings of the same mass dimensions d_i, the couplings of larger mass dimensions, and the couplings of one higher order mass dimensions $d_i - 2$, where $d_i < 0$. Therefore each β-function can be written in terms of the same order couplings, the quartic couplings, and one higher order couplings. This enables us to solve the β-functions in the following manner.

- First of all, we always solve for the coupling constants of the same mass dimensions simultaneously. For instance to solve for the coupling constants α_i and α_{i+1} where these have the same mass dimensions, we solve the equations

$\alpha_{i+2} = 0$ and $\alpha_{i+3} = 0$ simultaneously. [4]

- Then, by using this, we solve $\beta_i = 0$ and $\beta_{i+1} = 0$ simultaneously and we get a numerical value for the fixed points of the couplings α_i and α_{i+1}.

- We start from β_1 and β_2 and, input the numbers we find iteratively in the next order.

Since we are interested in the Veneziano limit where we have asymptotic safety for four dimensional gauge-Yukawa theories, we are only interested in the leading order behaviour in ϵ (equation (3.3)). The leading order of the fixed point values are given in table 3.1. We note that the higher order couplings could have changed the asymptotic safety of the theory, and could have an effect on the gauge-Yukawa interactions. But what we see in table 3.1 is that the theory is still asymptotically safe with the inclusion of higher order operators.

We find that the leading order terms in all the higher order coupling fixed points are increasingly higher order in ϵ. We observe that there is a general pattern in the increase in the power of ϵ at each iteration. ϵ's power increases by one with each iteration, with the only exception being the first iteration (i.e. from quartic order to sextic order). This fixed point picture in table 3.1 shows that since the higher order couplings are smaller and smaller, they maintain their perturbative properties.

Explanation for the absence of ϵ^2 term: For a generic coupling constant, we write the dimensionless coupling constant as $g_i = G_i k^{-d_i}$ where d_i is the canonical mass dimensions of the dimensionful coupling G_i. The running of this generic dimensionless coupling is defined with the β-function of the form

$$\beta_i = -d_i\, g_i + \# g_i^2 + O(g_i^3). \tag{3.39}$$

When the canonical mass dimension of the coupling constants is zero, because the linear term disappears, the leading order power of ϵ drops by half. Since the β-functions of the higher order couplings keep the linear piece, their power is not halved.

[4] We have checked that the higher order couplings contribute only to the higher order ϵ terms in the lower order couplings. This enables us to use this step.

Coupling	Fixed Point Value	Coupling	Fixed Point Value
α_1	$0.199781\,\epsilon$	α_2	$0.0625304\,\epsilon$
α_3	$0.442635\,\epsilon^3$	α_4	$0.197829\,\epsilon^3$
α_5	$-0.42182\,\epsilon^4$	α_6	$-0.0912196\,\epsilon^4$
α_7	$0.442354\,\epsilon^5$	α_8	$0.0561861\,\epsilon^5$
α_9	$-0.466105\,\epsilon^6$	α_{10}	$-0.0389432\,\epsilon^6$
α_{11}	$0.486798\,\epsilon^7$	α_{12}	$0.0287923\,\epsilon^7$
α_{13}	$-0.503072\,\epsilon^8$	α_{14}	$-0.0221745\,\epsilon^8$
α_{15}	$0.514813\,\epsilon^9$	α_{16}	$0.0175657\,\epsilon^9$
α_{17}	$-0.522305\,\epsilon^{10}$	α_{18}	$-0.0142047\,\epsilon^{10}$
α_{19}	$0.525963\,\epsilon^{11}$	α_{20}	$0.0116691\,\epsilon^{11}$
α_{21}	$-0.526232\,\epsilon^{12}$	α_{22}	$-0.00970593\,\epsilon^{12}$
α_{23}	$0.52355\,\epsilon^{13}$	α_{24}	$0.0081546\,\epsilon^{13}$
α_{25}	$-0.518328\,\epsilon^{14}$	α_{26}	$-0.00690831\,\epsilon^{14}$
α_{27}	$0.510945\,\epsilon^{15}$	α_{28}	$0.00589343\,\epsilon^{15}$
α_{29}	$-0.501742\,\epsilon^{16}$	α_{30}	$-0.00505756\,\epsilon^{16}$
α_{31}	$0.491027\,\epsilon^{17}$	α_{32}	$0.00436251\,\epsilon^{17}$
α_{33}	$-0.479076\,\epsilon^{18}$	α_{34}	$-0.00377977\,\epsilon^{18}$
α_{35}	$0.466132\,\epsilon^{19}$	α_{36}	$0.00328772\,\epsilon^{19}$

Table 3.1: Coupling constant fixed point values, given in their leading order in ϵ.

3.4.1 Critical exponents

As we introduced in the previous chapter, the critical exponents are universal quantities that determine the behaviour of the flow around the fixed point. In this section we find the scaling exponents which are defined as the minus of the eigenvalues of the stability matrix. The stability matrix is defined as $M_{ij} = \frac{\partial \beta_i}{\partial g_j}\Big|_*$ where g_j is a generic coupling constant and $*$ represents the evaluation at the fixed point. It may be a good point to remind the reader once again that in a predictive fundamental theory, it is expected that the operators should have a finite number of relevant operators. Also to remind that a negative eigenvalue we would have a relevant direction for the associated operator and, similarly, for a positive eigenvalue we have an irrelevant direction. According to this definition, if we have a β-function such as the one given in equation (3.39). We get the eigenvalue of the stability matrix in the form

$$\vartheta_i = -d_i + \text{non-Gaussian corrections.}^5 \tag{3.40}$$

Therefore, if the non-Gaussian corrections to the eigenvalues of the stability matrix are not dominant, the operators with negative mass dimensions will always have positive eigenvalues. This is called *the Bootstrap Hypothesis* shown to be satisfied for $f(R)$ quantum gravity up to 35 orders in $\mathcal{R}$ (Ricci scalar) [66, 67]. Therefore, however many higher dimensional operators we include in our current setup, they will be irrelevant operators because they have increasingly negative canonical mass dimensions.

We find that the eigenvalues of the stability matrix have increasingly positive ϵ corrections to their leading order values $(-d_i)$ with increasing order of operators, making the operators more and more irrelevant. We have shown that the Bootstrap Hypothesis is satisfied up to α_{36}. The figure 3.1 demonstrate this property. Each point is getting further away from the horizontal lines with increasing order of operators.

In order to demonstrate the change in the non-Gaussian part of the scaling exponents at each order, we define a variable $\tilde{\delta_n}$ and we plot it in figure 3.2. $\tilde{\delta_n}$ represents the non-Gaussian part of the scaling exponent with ϵ dependence taken

[5] Non-Gaussian corrections are the terms coming from the functional integration when we are computing the RHS of FRGE alongside with the contribution from the non-diagonal elements of the stability matrix.

θ_1	$4.03859\,\epsilon$	θ_2	$2.94059\,\epsilon$
θ_3	$2+4.24573\,\epsilon$	θ_4	$2+3.14773\,\epsilon$
θ_5	$4+5.29498\,\epsilon$	θ_6	$4+4.19698\,\epsilon$
θ_7	$6+6.34422\,\epsilon$	θ_8	$6+5.24622\,\epsilon$
θ_9	$8+7.39347\,\epsilon$	θ_{10}	$8+6.29546\,\epsilon$
θ_{11}	$10+8.44271\,\epsilon$	θ_{12}	$10+7.34471\,\epsilon$
θ_{13}	$12+9.49195\,\epsilon$	θ_{14}	$12+8.39395\,\epsilon$
θ_{15}	$14+10.5412\,\epsilon$	θ_{16}	$14+9.4432\,\epsilon$
θ_{17}	$16+11.5904\,\epsilon$	θ_{18}	$16+10.4924\,\epsilon$
θ_{19}	$18+12.6397\,\epsilon$	θ_{20}	$18+11.5417\,\epsilon$
θ_{21}	$20+13.6889\,\epsilon$	θ_{22}	$20+12.5909\,\epsilon$
θ_{23}	$22+14.7382\,\epsilon$	θ_{24}	$22+13.6402\,\epsilon$
θ_{25}	$24+15.7874\,\epsilon$	θ_{26}	$24+14.6894\,\epsilon$
θ_{27}	$26+16.8367\,\epsilon$	θ_{28}	$26+15.7387\,\epsilon$
θ_{29}	$28+17.8859\,\epsilon$	θ_{30}	$28+16.7879\,\epsilon$
θ_{31}	$30+18.9352\,\epsilon$	θ_{32}	$30+17.8371\,\epsilon$
θ_{33}	$32+19.9844\,\epsilon$	θ_{34}	$32+18.8864\,\epsilon$
θ_{35}	$34+21.0336\,\epsilon$	θ_{36}	$34+19.9356\,\epsilon$

Table 3.2: Numerical values for the eigenvalues of the stability matrix.

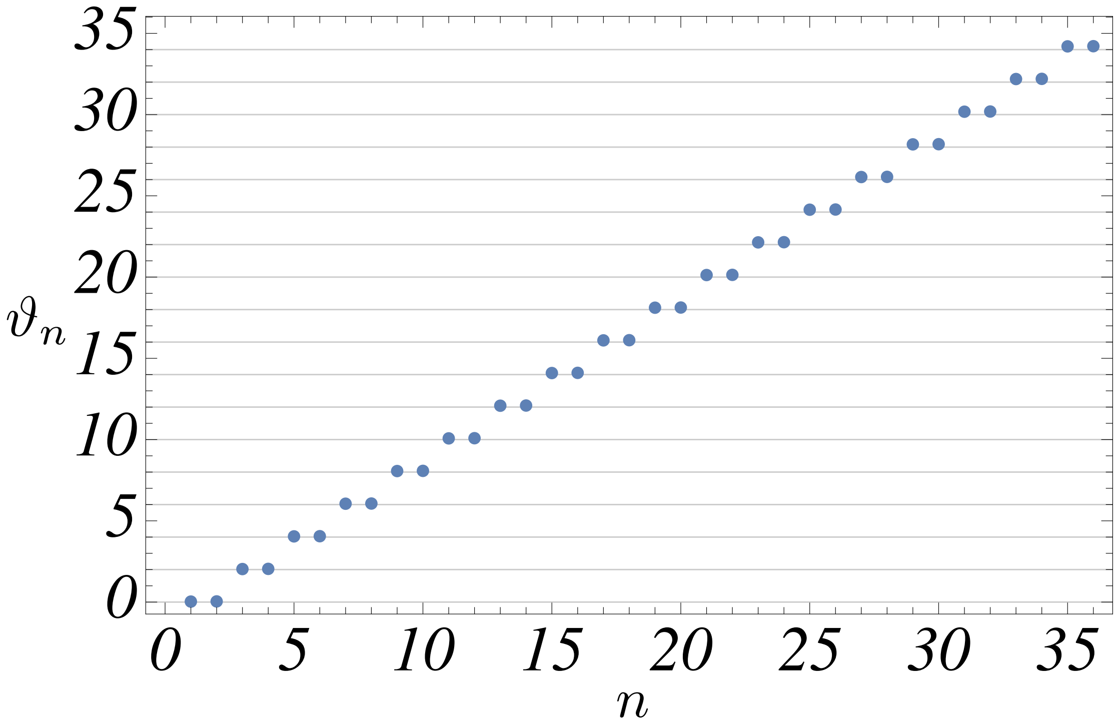

Figure 3.1: Eigenvalues of the stability matrix vs the order of the coupling α_n. Here we fix $\epsilon = 0.01$. Note that we have two points at each level due to the fact that one even and one odd coupling have the same canonical mass dimensions.

out, so that we don't have to numerically fix ϵ. Afterall we are interested in seeing whether this non-Gaussian bit is approaching zero therefore making the scaling exponent "very" Gaussian. We define $\tilde{\delta_n}$ such as

$$\vartheta_n = -d_n\big(1 + \Delta_n\epsilon + O(\epsilon^2)\big) \tag{3.41}$$

$$\delta_n \equiv \Delta_n\epsilon + O(\epsilon^2) \tag{3.42}$$

$$\tilde{\delta_n} \equiv \delta_n/\epsilon \tag{3.43}$$

For the values we find in table 3.2, $\tilde{\delta_n}$ can be computed for each truncation order, n. We have shown in figure 3.2 that the value of $\tilde{\delta_n}$ gets smaller and smaller with the inclusion of more and more higher order operators. This is another way of showing that the Bootstrap Hypothesis is satisfied. Since the values of the higher order scaling exponents stay near-Gaussian, we can rest assured that the higher order operators will stay irrelevant regardless of how many higher order operators we include in the truncation.

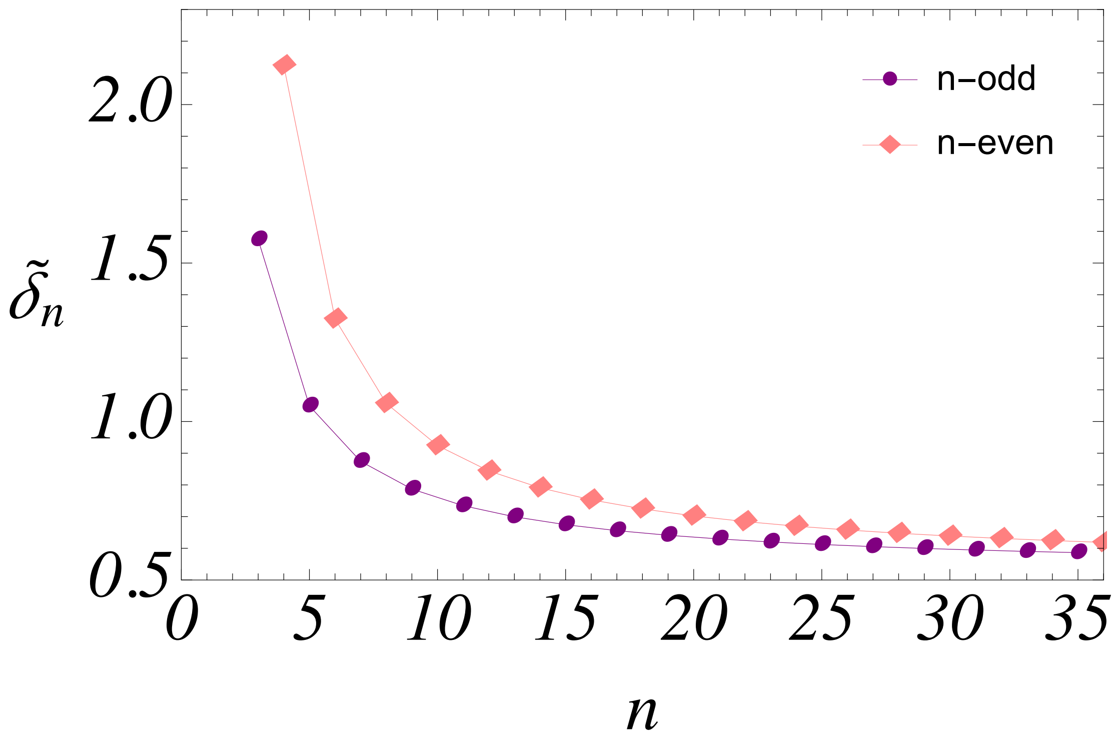

Figure 3.2: Shown is the fast approach to near-Gaussian scaling with increasing canonical mass dimension for all scaling exponents ϑ_n related to higher dimensional interactions. ϵ is fixed to the value 0.01.

3.5 Analytic solution in the leading order

In this section we show that an analytical closed expression can be obtained for the potential. This is achieved by using a recursive relation for the coupling fixed point values for arbitrarily higher order scalar self-interactions to leading order in ϵ. This is the same as saying that the results given are leading order in α_y^*, the Yukawa coupling fixed point, because the ϵ contribution comes from the Yukawa coupling fixed point value.

We first set ρ_2-type interactions to zero. We only compute the u_k part of the potential. Due to reasons mentioned in section 3.3, we are able to separate the potential and this makes the computation easier. The flow for the u_k part of the potential is given in equation (3.31). We write this equation again without the Yukawa contribution, since we know that the higher order interactions do not get a contribution from the Yukawa or gauge interactions. Here we are interested in computing only the higher dimensional terms in the potential, starting from the sextic interactions. Therefore we take the quartic and Yukawa coupling fixed points as given.

$$\partial_k u_k = -4u_k + 2\rho_1 u_k' + \frac{1}{2}\left(\frac{1}{1+u_k'} + \frac{1}{1+u_k'+4\rho_1 c_k}\right). \tag{3.44}$$

As we noted before, even though we switched off the ρ_2-type interactions, the c_k part of the potential still appears in the flow of u_k.

To compute the β-functions of the nth order coupling constant, we take the nth derivative of the flow equation with respect to the field ρ_1 evaluated at $\rho_1 = 0$. Taking the arbitrary nth order derivative will normally bring a lot of higher order derivatives of u_k and c_k. We make the observation that these terms are higher order in the ϵ expansion and we can ignore these terms to compute the fixed points at the leading order in ϵ. From our numerical results we see that higher order couplings are higher order in ϵ. For example, the sextic order coupling constants are $O(\epsilon^3)$, eighth order coupling constants are $O(\epsilon^4)$, and so on. Therefore to compute the nth order fixed point we can ignore the $O(\epsilon^{\frac{n}{2}+1})$ terms.

We find the nth derivative of the flow evaluated at $\rho_1 = 0$ is given by

$$\partial_t u_k^{(n)} = -4u_k^{(n)} + 2nu_k^{(n)} + \frac{1}{2}(-1)^n n!\big((u_k'')^n + (u_k''+4c_k)^n\big). \tag{3.45}$$

Note that $u_k^{(n)} = \frac{\partial^n u_k}{\partial \rho_1^n}$, the nth derivative with respect to ρ_1 where $n \geq 3$ and u_k'' denotes the second derivative with respect to ρ_1. At $\rho_1 = 0$, $u_k'' = 2\alpha_2$ and $c_k = \alpha_1$. Therefore from above equation, we can find a recursive relation for the even-numbered fixed points as the following.[6]

$$\alpha_{2n-2} = \frac{u_k^{(n)}}{n!} = \frac{-1}{2(2n-4)}(-1)^n\left((2\alpha_2)^n + (2\alpha_2+4\alpha_1)^n\right) \tag{3.46}$$

where $n \geq 3$.

We remind the reader again what we discussed in section 3.2. The even-numbered couplings describe the interactions that are defined in u_k part of the potential whereas the odd-numbered couplings describe the interactions that are defined in

[6] We drop the asterix, *, for the fixed points, but the reader should note in this section that by α_1, α_2, and α_y, we mean their fixed point values. These values should be known in order to compute the higher order couplings.

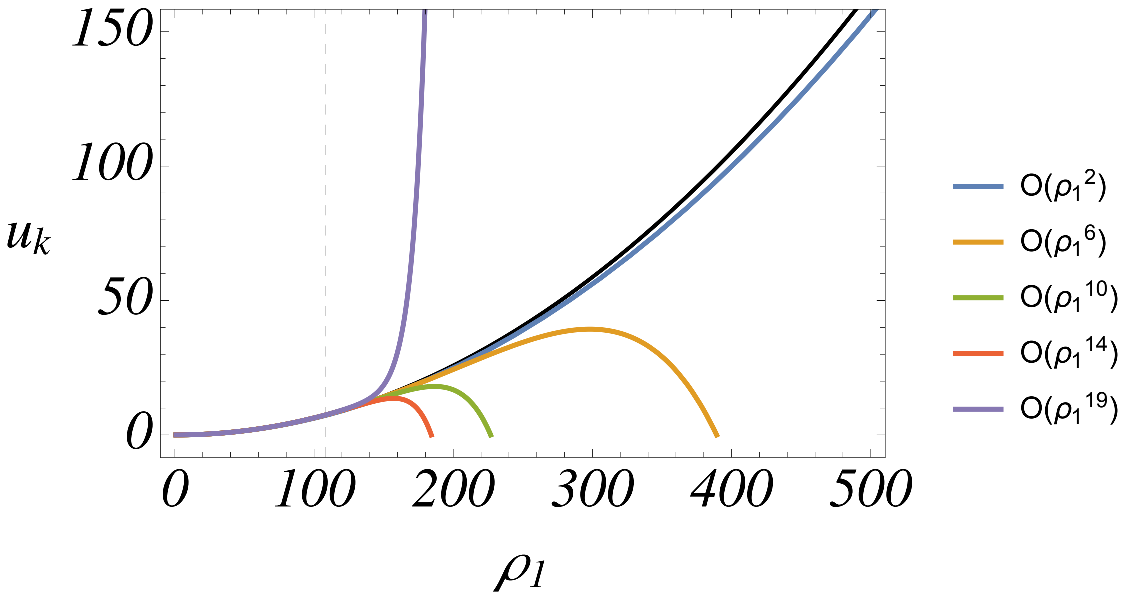

Figure 3.3: The u_k part of the potential is plotted against the ρ_1 field values. $\rho_2 = 0$ and $\epsilon = 0.01$ where the black curve represents the exact resummed potential. The coloured curves are the potential with given truncations. Vertical dashed line marks the radius of convergence.

c_k. Then the u_k part of the potential can be written as a sum.

$$u_k(\rho_1) = \alpha_2\rho_1^2 + \sum_{n=3}^{\infty} \alpha_{2n-2}\rho_1^n \tag{3.47}$$

Although it would be equivalent to write the quartic term in the sum and starting the sum from $n = 2$, we write it separate to the summation. This is because we resum the part expressed with the summation symbol, by using equation (3.45) which is an expression for only $n \geq 3$. By plugging the equation (3.45) in equation (3.47) can resum the potential. We find

$$u_k(\rho_1) = \alpha_2\rho_1^2 + 4\rho_1^2\left[\alpha_2^2\ln(1+2\alpha_2\rho_1) + (2\alpha_1+\alpha_2)^2\ln(1+(2\alpha_2+4\alpha_1)\rho_1)\right] \tag{3.48}$$

We see that this is a Coleman-Weinberg type potential [68] in the large-N approximation, with high order couplings bringing in a logarithmic correction to the quartic potential. We plot this resummed potential and compare the exact equation with the numerical solutions that we found above in figure 3.3.

Now if we switch on ρ_2-type interactions, we compute the flow for the $c_k(\rho_1)$ part of the potential as well. We redefine the repeating coefficients to simplify the equations.

$$A \equiv 2\alpha_2\,, \tag{3.49}$$

$$B \equiv 4\alpha_1 + 2\alpha_2\,, \tag{3.50}$$

$$C \equiv B - A = 4\alpha_1\,. \tag{3.51}$$

Similarly, the odd-numbered β-functions and fixed points will be found by taking appropriate derivatives of the flow of the c_k part of the potential given in equation (3.32). We remind the reader once again that, since the higher order couplings do not get contribution from the Yukawa interactions directly, we do not include the Yukawa contribution in the flow equation. This part of the equation disappears once the appropriate derivatives are taken, in order to compute the β-function. We simplify the rest of the equation further by eliminating the contribution from the higher order ϵ terms. Since we know from the numerical computation that c_k' is $\mathcal{O}(\epsilon^3)$ and multiplications of c_k' with other terms will bring even higher order ϵ terms. Therefore we do not include them in our analytical computation, as we are interested in the leading order ϵ behaviour here. Under these considerations, the flow of c_k is given as

$$\partial_t c_k = 2\rho_1 c_k' + \frac{1}{2}\left(c_0 + c_2 + c_3\right), \tag{3.52}$$

$$c_0 = \frac{16c_k^2}{(1+u_k'+4\rho_1 c_k)^3}, \tag{3.53}$$

$$c_2 = \frac{64c_k^4\rho_1^2}{(1+u_k')^2(1+u_k'+4\rho_1 c_k)^3}, \tag{3.54}$$

$$c_3 = \frac{128c_k^5\rho_1^3}{(1+u_k')^3(1+u_k'+4\rho_1 c_k)^3}. \tag{3.55}$$

Here we have several multiplications of functions of ρ_1. And we need to take the nth derivative of these products in order to find the β-functions. In order to accomplish this, we use the general formula for the nth derivative of a product of two functions given as

$$\partial_x^n(f \times g) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}, \tag{3.56}$$

where f and g are functions of the variable x. Then we apply this formula to take the nth derivative of the flow of c_k. Then the leading order terms are found as

$$\partial_t c_k^{(n)} = 2nc_k^{(n)} + \frac{1}{2}(c_0^{(n)} + c_2^{(n)} + c_3^{(n)}), \tag{3.57}$$

$$c_0^{(n)} = \frac{16}{2}\alpha_1^2(-1)^n(n+2)!B^n, \tag{3.58}$$

$$c_2^{(n)} = 64\alpha_1^4 n(n-1)(-1)^{n-2}\sum_{k=0}^{n-2}\frac{1}{2}(k+1)!A^k(n-k)!\binom{n-2}{k}B^{-k+n-2}, \tag{3.59}$$

$$\begin{aligned} c_3^{(n)} &= 128\alpha_1^5 n(n-1)(n-2)(-1)^{n-3} \\ &\times \sum_{k=0}^{n-3}\frac{1}{4}(k+2)!A^k(-k+n-1)!\binom{n-3}{k}B^{-k+n-3}. \end{aligned} \tag{3.60}$$

Upon rearranging the above equation, we find the odd-numbered coupling fixed point with the expression

$$\alpha_{2n+1} = \frac{-1}{(4n)n!}\left(c_0^{(n)} + c_2^{(n)} + c_3^{(n)}\right) \tag{3.61}$$

When we insert all the terms in the above expression, we observe that we can perform a resummation. The resummed equations are given as

$$\sum_n -\frac{1}{4nn!}c_0^{(n)}\rho^n = c_0^{\text{res}} = \frac{C^2B(3B\rho_1+4)\rho_1}{8(1+B\rho_1)^2} + \frac{C^2}{8}\ln(1+B\rho_1)\,, \tag{3.62}$$

$$\begin{aligned} \sum_n -\frac{1}{4nn!}c_2^{(n)}\rho^n = c_2^{\text{res}} &= -\frac{CA(A+2B)\rho_1}{16(1+A\rho_1)(1+B\rho_1)^2} - \frac{CB\left(8A^2+3AB+B^2\right)\rho_1^2}{32(1+A\rho_1)(1+B\rho_1)^2} \\ &- \frac{CAB^2(5A+B)\rho_1^3}{32(1+A\rho_1)(1+B\rho_1)} + \frac{A}{16}(A+2B)\ln\frac{1+B\rho_1}{1+A\rho_1}\,, \end{aligned} \tag{3.63}$$

$$\begin{aligned} \sum_n -\frac{1}{4nn!}c_3^{(n)}\rho^n = c_3^{\text{res}} &= \frac{C\left(A^2+4AB+B^2\right)\rho_1}{32(1+A\rho_1)^2(1+B\rho_1)^2} + \frac{3C\left(A^3+5A^2B+5AB^2+B^3\right)\rho_1^2}{64(1+A\rho_1)^2(1+B\rho_1)^2} \\ &+ \frac{CAB\left(5A^2+8AB+5B^2\right)\rho_1^3}{32(1+A\rho_1)^2(1+B\rho_1)^2} + \frac{3CA^2B^2\left(A+B\right)\rho_1^4}{32(1+A\rho_1)^2(1+B\rho_1)^2} \\ &+ \frac{1}{32}\left(A^2+4AB+B^2\right)\ln\frac{1+A\rho_1}{1+B\rho_1}. \end{aligned} \tag{3.64}$$

Then, combining equation (3.61) with equations (3.62)-(3.64) we can write the overall summed and simplified α_{2n+1} as

$$\begin{aligned} \sum_{n=1}^{\infty}\alpha_{2n+1}\rho^n = \alpha_{2n+1} &= \frac{C^2(A+17B)\rho_1}{32(1+A\rho_1)^2(1+B\rho_1)^2} + \frac{C^2\left(A^2+74AB+25B^2\right)\rho_1^2}{64(1+A\rho_1)^2(1+B\rho_1)^2} \\ &+ \frac{ABC^2(19A+27B)\rho_1^3}{32(1+A\rho_1)^2(1+B\rho_1)^2} + \frac{7A^2B^2C^2}{16(1+A\rho_1)^2(1+B\rho_1)^2} \\ &+ (A+B)\ln(1+A\rho_1) + (7C-2A)\ln(1+B\rho_1) \end{aligned} \tag{3.65}$$

Finally, we can use the fact that

$$c_k(\rho_1) = \alpha_1 + \sum_{n=1}^{\infty} \alpha_{2n+1}\rho_1^n, \tag{3.66}$$

and the expression for u_k given above in equation (3.48) to obtain a full solution for the potential up to arbitrarily high orders in ρ_1 in the leading order in ϵ. Here, similar to the above resummation for u_k, we do not include the quartic term in the sum by starting the sum from $n = 0$. This is due to the fact that we resum the expression given in the sum symbol.

Recall that the potential is defined as $v_k(\rho_1, \rho_2) = u_k(\rho_1) + \rho_2 c_k(\rho_1)$. Now we have all parts of this potential analytically. We have shown that the full potential can be resummed and that overall contribution from all the higher order scalar self couplings of type ρ_1^n and ρ_2 are included in this potential (all in leading order in ϵ).

This result is extremely important in our study, as we can find the β-function or the fixed point values of an arbitrary order coupling in the truncation. Therefore we know the effect of an arbitrarily high order operator to the asymptotic safety of four-dimensional gauge-Yukawa theories.

3.6 Universality

A careful reader might have noticed by looking at figure 3.1 or table 3.2 that there is an interesting behaviour in the scaling exponents. To explain this behaviour, let us first of all differentiate between the odd and even-numbered ϑ's as ϑ_{2n+1} and ϑ_{2n+2} where $n = 0, 1, 2, \cdots$. ϑ_{2n+1} are the eigenvalues of the stability matrix where α_{2n+1} dominates which describe the $c_k(\rho_1)$ part of the potential that is linear in ρ_2, whereas ϑ_{2n+2} are the eigenvalues of the stability matrix where α_{2n+2} dominates. These even-numbered coupling describe the $u_k(\rho_1)$ part of the potential.

The interesting behaviour that we talk about is that the scaling exponents have a fixed difference between the eigenvalues of the stability matrix. The observed rule given as

$$\vartheta_1 + (2 + 0.207139\epsilon) = \vartheta_3, \tag{3.67}$$

$$\vartheta_2 + (2 + 0.207139\epsilon) = \vartheta_4, \tag{3.68}$$

$$\vartheta_{2n+1} + (2 + 1.04924\epsilon) = \vartheta_{2n+3}, \tag{3.69}$$

$$\vartheta_{2n+2} + (2 + 1.04924\epsilon) = \vartheta_{2n+4} \qquad \forall n \geq 1. \tag{3.70}$$

Let us define

$$\vartheta_i = -d_i + a_i\epsilon, \tag{3.71}$$

up to $\mathcal{O}(\epsilon)$. Since the mass dimension will decrease by 2 in the next step, we have

$$\vartheta_{i+2} = -(d_i + 2) + a_{i+2}\epsilon, \tag{3.72}$$

up to $\mathcal{O}(\epsilon)$. Then, given equations (3.67) - (3.70), for $\forall i \geq 3$, the difference between the $\mathcal{O}(\epsilon)$ terms are found as

$$a_{i+2} - a_i = 1.04924, \tag{3.73}$$

and for $i = 1, 2$

$$a_{i+2} - a_i = 0.207139. \tag{3.74}$$

In order to understand the reason behind this, we look at the stability matrix itself. A part of the stability matrix $\frac{\partial \beta_i}{\partial \alpha_j}\Big|_*$ is shown below

$$\begin{pmatrix} 4.03859\epsilon & 0 & -1 & 0 & 0 & 0 \\ 3.69673\epsilon & 2.94059\epsilon & -2 & -3 & 0 & 0 \\ 0 & 0 & 2+4.24573\epsilon & 0 & -2 & 0 \\ 0 & 0 & 3.69673\epsilon & 2+3.14773\epsilon & -2 & -4 \\ 0 & 0 & 0 & 0 & 4+5.29498\epsilon & 0 \\ 0 & 0 & 0 & 0 & 3.69673\epsilon & 4+4.19698\epsilon \end{pmatrix} \tag{3.75}$$

We see that the $\mathcal{O}(\epsilon)$ correction is already there in the diagonal term. Next we show where these $\mathcal{O}(\epsilon)$ terms are coming from. We show this only for the even-numbered couplings which belong in u_k part of the potential, since it is easier to show. A similar calculation can be done for the c_k part, though this would be more complicated for the purposes of this study.

In equation (3.45), we show only the leading order terms as we explained. If a derivative of (3.45) with respect to $u^{(n)}$ is taken, this gives the canonical mass dimension, which is the leading order (integer) term of the scaling exponents. However, we wish to investigate the next-to-leading order ($\mathcal{O}(\epsilon)$) terms that appear in the diagonal of the stability matrix. Therefore we must look into equation (3.44).

Since the rest of the equation can only give rise to the leading order term, the culprit in (3.44) that gives the $\mathcal{O}(\epsilon)$ term in the scaling exponents is the following.

$$\frac{1}{2}\left(\frac{1}{1+u_k'(\rho_1)}+\frac{1}{1+u_k'(\rho_1)+4\rho_1 c_k(\rho_1)}\right). \tag{3.76}$$

If we take n-derivatives of this term with respect to the field ρ_1, we will get its contribution to the β-function of α_{2n-2}. And we are looking to get the linear term to α_{2n-2}. For the nth derivative we get

$$\partial_t u_k^{(n)} \supset \frac{1}{2}\left(2nu_k''+2n(u_k''+4c_k)+(4-2n)\right)u_k^{(n)}, \tag{3.77}$$

$$\implies \beta_{2n-2} \supset \left(4n(\alpha_2+\alpha_1)+(4-2n)\right)\alpha_{2n-2}, \tag{3.78}$$

where the $\supset$ symbol means that we do not include all the terms in the equation as here we are only interested in the term linear to α_{2n-2}. Therefore there is a fixed difference in scaling exponents in each order. Because in each order the coefficients of α_1 and α_2 increases by 4. From equation (3.78), we can write the associated (same order) diagonal element in the stability matrix as,

$$\vartheta_{2n-2} = \underbrace{(4-2n)}_{\text{Leading order}} + \underbrace{4n(\alpha_2+\alpha_1)}_{\mathcal{O}(\epsilon)\text{ term}}, \tag{3.79}$$

where we note that $(4-2n)$ is the canonical mass dimension of the coupling α_{2n-2}. Given the values of n at each level of iteration, this can numerically be shown that it corresponds exactly at the difference between the scaling exponents of the even-numbered couplings. A similar equation can be shown for the odd-numbered couplings.

Next we show that the $\mathcal{O}(\epsilon)$ term in scaling exponents are universal, by showing their independency from the regulator. For this we need to look into the functional integral where the regulator R_k appears. Recall that we redefined everything to be dimensionless, so we use $r(y)$ as a regulator which is equivalent to $R_k(q)$. Recall the necessary properties of R_k explained in section (3.3). These translate to the redefined regulator $r(y)$ to behave as the following.

$$\text{As} \begin{cases} y \to \infty, & r(y) \to 0, \\ y \to 0, & r(y) \to \infty. \end{cases} \tag{3.80}$$

The part of the functional integral that the $\mathcal{O}(\epsilon)$ term in the scaling exponents comes from is given as

$$\begin{aligned}
\vartheta_{2n-2} = \frac{\partial}{\partial u_k^{(n)}} \partial_t u_k^{(n)} &= -d_{2n-2} + \frac{1}{2}\int_0^\infty \mathrm{d}y y \left(\frac{-2y^2 r' 2n u_k''}{(y(1+r))^3} + \frac{-2y^2 r' 2n (u_k'' + 4c_k)}{(y(1+r))^3} \right), \\
&= -d_{2n-2} + \left(\frac{4n\alpha_2}{2} + \frac{4n\alpha_2 + 8n\alpha_1}{2} \right) \int_0^\infty \mathrm{d}y \frac{-2r'}{(1+r)^3}, \\
&= -d_{2n-2} + (2n\alpha_2 + (2n\alpha_2 + 4n\alpha_1)) \left[\frac{1}{(1+r)^2} \right]_{y=0}^{y=\infty}, \\
&= -d_{2n-2} + 4n\left(\alpha_2 + \alpha_1\right),
\end{aligned} \tag{3.81}$$

which is evaluated at the origin where $u_k' = 0$ and $\rho_1 = 0$. Also, d_{2n-2} denotes the canonical mass dimensions of the coupling constant α_{2n-2}.

Irrelevant of the choice of the regulator, by only using the properties that all regulators should have in order to satisfy the EAA, we get the $\mathcal{O}(\epsilon)$ terms as we found numerically. Therefore not only the leading order term, but also the $\mathcal{O}(\epsilon)$ term is universal.

3.7 Radius of convergence

For the series expansion of the potential in our theory, which is evaluated at the fixed point, we can find the radius of convergence analytically as we know the fixed point values of an arbitrary order coupling in the leading order in ϵ. It is important that we have a finite radius of convergence as the validity of any truncations of this potential is dependent on this.

Now that we have a closed equation for the potential, we can compute the radius of convergence of the series that is the fixed point potential. We use the root test to compute the radius of convergence. Root test is given as

$$r = \lim_{n\to\infty} |a_n|^{-1/n}, \tag{3.82}$$

where r is the radius of convergence and a_n is the coefficient of the nth order term a given series. In our case this is the coupling constant.

Let us, once again, look at the radius of convergence of the expansion of only the u_k part of the potential. The closed form equation for the coupling constants was found in equation (3.46). We apply the root test given in (3.82), and we find that the radius of convergence is

$$r \approx \frac{19}{2\left(\sqrt{20+6\sqrt{23}}+\sqrt{23}-3\right)\epsilon} = \frac{1.08204}{\epsilon} \tag{3.83}$$

This is exactly equal to $\frac{1}{B}$ where B was defined in equation (3.50). In other words,

$$r = \frac{1}{B}. \tag{3.84}$$

This value is shown in figure 3.3 as dashed vertical line.

For a given finite truncation, if we are looking at a field value of $\rho_1 > 1/B$, the truncation is not reliable. Having said that, since we have the resummed expression for the potential up to leading order in ϵ, for the values of the field $\rho_1 > 1/B$, we can rely on the resummed potential.

3.8 Discussion

In this chapter we investigated the effects of higher order scalar operators on four dimensional gauge-Yukawa theories that has recently been shown to be asymptotically safe in reference [22]. Adding these higher order interactions could have a devastating effect on the asymptotic safety of the said theory. However with the hind-sight, we can safely say that with the higher order interactions that we added to the theory, asymptotic safety is still there, with all except one irrelevant operators.

We started by numerically investigating the fixed points of these beyond-marginal-mass-dimensional coupling constants. We first find that the leading order terms in higher order coupling fixed points are higher order in ϵ. This means that the contribution of higher and higher order operators is smaller and smaller. In that sense this is a perturbative expansion. Since the value of ϵ needs to be small in the Veneziano limit, the higher order couplings are weakly coupled. However the theory still has an interacting fixed point with the inclusion of higher dimensional scalar self interactions. So we can confirm that the higher order operators do not destroy asymptotic safety.

We find that the eigenvalues of the stability matrix are in the form $(-d_i)+\mathcal{O}(\epsilon)$. This means that they are very close to Gaussian, as can be seen in figure 3.1. We also confirm that this satisfies the Bootstrap Hypothesis as higher order operators are more and more irrelevant. This means that as we add more and more higher order terms, they will be irrelevant.

We find that the β-functions and the fixed points can be computed analytically, directly from the Wetterich equation, in their leading order in ϵ. Remarkably, we find a closed expression for the coupling constants of higher order in terms of the quartic couplings. So we know the value of the fixed point of an arbitrary order of coupling, its associated scaling exponent, and the value for radius of convergence of the potential up to $\mathcal{O}(\epsilon)$. In addition to this, we show that the next-to leading order term in the scaling exponents are universal just like the leading order term, or in other words, independent of the choice of the regulator R_k.

To gain further insight into this theory following items can be a topic of further investigation.

- Contribution of higher order combinations of ρ_2 and ρ_1, as well as higher order Yukawa interactions may be investigated. It is important to check that none of these contributions have an effect on the asymptotic safety of the theory. It is also essential to check that the scaling exponents still satisfy the Bootstrap Hypothesis with even more non-trivial operators are included.

- Investigating the effect of a mass term and symmetry breaking, and also the anomalous dimension when the wave function renormalisation factors $Z_Q \neq 1 \neq Z_H$, may give us more insight about the UV properties of this theory. This then might give us clues about the beyond the standard model theories.

- Cosmological applications of this theory might give us indications about inflation theories. Since we are able to compute the ressumed analytical potential at large field values, a further study using this potential can check whether our theory is compatible with viable inflation theories.

Overall, both numerically and analytically we show that the coupling constants of this theory run to a fixed point. All the operators in our theory have irrelevant directions except for the gauge coupling, which has a relevant direction. This is a good indicative of a predictive and fundamental theory.

Chapter 4

Quantum gravity in the large-D limit

At time of writing of this thesis, quantum gravity remains to be one of the biggest open questions in physics. Although we are interested in looking at it from asymptotic safety perspective which was first introduced in [6] and investigating the UV behaviour, there are many other popular methods to address the question of a quantum gravity. Some of these methods are based on redefining a discrete/quantised space-time such as loop quantum gravity [69, 70] and as an application to it loop quantum cosmology [71], or quantising only the space such as causal sets [72], or causal dynamical triangulations [73]. Some of the methods rely on using the well trusted quantum field theory such as the effective field theory [74].

In a theory of gravity, the parameter analogous to the number of group degrees of freedom, N, is given by the number of space-time dimensions, D. Hence it would seem promising to investigate whether a large-D limit of quantum gravity may lead to simplifications similar to large-N limit [25]. Furthermore, duality arguments in string theory, such as [75], suggest a deeper link between gravity and gauge theories.

From a phenomenological point of view, models such as the ADD model [35] tell us that the Planck scale takes much smaller values when compact extra dimensions are included. Since we know that quantum gravity effects become significant at the Planck scale, this would mean that quantum gravity effects could be observed at energy scales within reach of current accelerators.

In quantum gravity, the idea of large dimension expansion was first followed by

Strominger [34] by studying the leading ultraviolet momentum behaviour of specific (classes of) Feynman diagrams where it has been suggested that gravity simplifies in both non-compact and highly compactified extra dimensions cases. With the help of a rescaling of the gravitational coupling, it has been shown that two loops cannot share the same propagator. Hence disjointed bubble graphs are preferred over the nested ones.

Recently this line of research has also been followed within an effective field theory approach [76]. A very similar result to the previous one has been found in the effective field theory case in that a subset of planar diagrams carry the leading $1/D$ contributions to the n-point Green's Functions. Thus a large-D limit of a given n-point function will consist of bubble and (in contrast to the previous result) vertex-loop graphs with the nested graphs ignored due to their being higher order in $1/D$.

Similarly for black holes it has been shown that quantum gravity simplifies even within classical general relativity theory such that, in the large-D limit, interactions between the black holes becomes negligible [77, 78, 79].

Most recently, in [80], cosmological implications of a D-dimensional quantum gravity on an N-torus[1] in the Kaluza-Klein construction have been investigated where $N \to \infty$. It has been discussed that although it is very unlikely that the physical universe has infinitely many dimensions, the large number of extra dimensions are shown to stabilise the vacuum solutions.

The most important difference between standard large-N expansions and a large D expansion is transparent in these works. That is in the planar limit of QCD, the parametric dependence on N of the leading Feynman diagrams originates solely from the contraction of group factors. In turn, in quantum gravity the parameter D not only appears as a pre-factor from group factors, but also from the loop integration itself. Clearly, this reflects the fact that an expansion in $1/D$ is an expansion in every loop order and group factor at the same time. On the other hand, very similar to large-N theories, our aim is to simplify quantum gravity by using a large-D limit.

Here, we access the large dimension limit by means of an functional renormalisation group. For quantum gravity, this approach has been made available in [10].

[1] Here N-torus chosen as the shape of the background field.

The approach keeps the dimension D as a continuous parameter and allows for a systematic study of a large-D limit.

In this chapter, we study the fixed points of quantum gravity in the limit of large dimensions. We restrict ourselves to the Einstein-Hilbert truncation, which retains the volume element and the Ricci scalar as independent operators. So we look into two coupling constants, i.e. the gravitational constant and the cosmological constant. As a result, we find that physical non-trivial fixed points exist in a large number of dimensions and, from several different approximations, give us a clear and stable picture of the 1/D expansion of quantum gravity. It is important to note that this expansion of fixed points in $1/D$ has a finite radius of convergence with real positive scaling exponents which are gauge independent.

4.1 Recap: Advances in asymptotically safe gravity

In this section we give our setup and review some of the results obtained using the functional renormalisation group method for quantum gravity. We are interested in the so-called Einstein-Hilbert truncation where we only retain the minimal amount of operators, i.e. the Ricci scalar and the cosmological constant.

The flow equations that are computed using an Einstein-Hilbert truncation have been studied in the literature immensely. Since quantum gravity is perturbatively non-renormalisable due to the gravitational coupling having negative mass dimensions, the functional renormalisation group method is the perfect candidate to explore a predictive quantum field theory of gravity. A large number of previous studies have established the existence of a UV fixed point for the Einstein-Hilbert truncation that has finite number of relevant directions [10, 81, 82, 11, 83, 19, 14]. The non-trivial IR fixed point is also investigated in references [84, 85, 86, 45]. Some extensions to Einstein-Hilbert truncation, including higher derivative expansion and $f(R)$ expansion [87, 88, 89, 90, 91, 89, 66, 67] and fluctuations in the ghost sector [92, 93, 94], have been addressed. There also has been many important advances in coupling gravity with matter fields [81, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106]. Cosmological applications of quantum gravity from asymptotic safety perspective have also been studied [107, 108, 109, 110, 111].

Here we are mainly focusing on the *background field technique*, otherwise known as the *single metric truncation*, which uses a decomposition of the metric field according to York decomposition [112]. We also compare our results with the *bimetric truncation* which aims to remove background independence by separating the metric into a background and a dynamical piece [113]. The details of the latter method will be given in the later sections. Now we explain the background field technique. We start with the Einstein-Hilbert truncation. When the couplings are running with the energy scale, the action becomes the scale-dependent EAA given as

$$\Gamma_k = -\frac{1}{16\pi G_k}\int \mathrm{d}^D x\sqrt{g}\,[R(g) - 2\Lambda_k] + S_{\text{gauge}} + S_{\text{ghost}} \tag{4.1}$$

where g is the determinant of the metric $g_{\mu\nu}$, $R(g)$ is the Ricci scalar, G_k is the scale dependent gravitational constant and Λ_k is the scale dependent cosmological constants. Also $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ is the background part of the metric and $h_{\mu\nu}$ is the dynamical part of the metric. Here we also have the gauge and the ghost actions described by S_{gauge} and S_{ghost}, respectively.

Gauge fixing: Here we follow [10, 11], and we write the gauge fixing action as

$$S_{\text{gauge}} = \frac{1}{2\alpha}\int \mathrm{d}^D x\sqrt{\bar{g}}F_\mu F^\mu. \tag{4.2}$$

where we choose the gauge condition $F_\mu = \frac{1}{\sqrt{16\pi G_k}}\left(\bar{\nabla}^\nu h_{\mu\nu} - \frac{1+\rho}{D}\bar{\nabla}_\mu h^\nu{}_\nu\right)$, i.e. the harmonic gauge. Note that the bar on the symbols mean that they belong to the background part of the metric.

We choose a spherically symmetric background to simplify our equations by enabling us to write the Riemann tensor, and Ricci tensor as proportional to the Ricci scalar such as

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{\bar{R}}{D(D-1)}\left(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\right), \tag{4.3}$$

$$\bar{R}_{\mu\nu} = \frac{\bar{R}}{D}\bar{g}_{\mu\nu}, \tag{4.4}$$

where $\bar{R}_{\mu\nu\rho\sigma}$ is the Riemann tensor, $\bar{R}_{\mu\nu}$ is the Ricci tensor where $\bar{R}_{\mu\nu} = \bar{R}^\sigma_{\mu\sigma\nu}$, and $\bar{R} = \bar{R}^\mu{}_\mu$. We note that throughout this chapter, the quantities that have a bar sign on them describes the background. Note that D is the number of space-time dimensions, inverse of which we will use as an expansion parameter in the next section.

In the background field method we write the metric as

$$g_{\mu\nu} = h_{\mu\nu} + \bar{g}_{\mu\nu} \tag{4.5}$$

where $\bar{g}_{\mu\nu}$ is the background metric and $h_{\mu\nu}$ is the dynamical metric that defines the graviton field. York decomposition is based on a transverse-traceless (TT) decomposition of the metric, where we decompose the dynamical part of the metric into a transverse-traceless part, $h^{\mathrm{T}}_{\mu\nu}$, a longitudinal-transverse part, $h^{\mathrm{LT}}_{\mu\nu}$, a longitudinal-longitudinal part, $h^{\mathrm{L}}_{\mu\nu}$, and a trace part, $h^{\mathrm{Tr}}_{\mu\nu}$. These are defined as

$$h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} + h^{\mathrm{LT}}_{\mu\nu} + h^{\mathrm{L}}_{\mu\nu} + h^{\mathrm{Tr}}_{\mu\nu}, \tag{4.6}$$

$$h^{\mathrm{LT}}_{\mu\nu} = \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \xi_\nu, \tag{4.7}$$

$$h^{\mathrm{L}}_{\mu\nu} = \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma - \frac{1}{D}\bar{g}_{\mu\nu}\bar{\nabla}^2\sigma, \tag{4.8}$$

$$h^{\mathrm{Tr}}_{\mu\nu} = \frac{1}{D}\bar{g}_{\mu\nu} h, \tag{4.9}$$

where

$$h^{\mathrm{T}}_{\mu\nu} = h^{\mathrm{T}}_{\nu\mu}, \tag{4.10}$$

$$h^{\mathrm{T}\mu}{}_{\mu} = 0, \tag{4.11}$$

$$\bar{\nabla}_\nu h^{\mathrm{T}\nu}{}_{\mu} = 0, \tag{4.12}$$

$$\bar{\nabla}^\nu \xi_\nu = 0, \tag{4.13}$$

$$h = \bar{g}_{\mu\nu} h^{\mu\nu} \tag{4.14}$$

Note that when we compute the functional integral from (2.29), according to this decomposition, in addition to the transverse-traceless part, we will have contributions from the eigenvalues of transverse vector ξ_ν and scalars σ and h, although some of the modes do not contribute to the functional integral. These are the modes corresponding to two lowest eigenvalues of σ and the lowest eigenvalue of the transverse vector ξ_ν. These are not physical because they satisfy the Killing equation, giving $h_{\mu\nu} = 0$ [11, 88]. So they will be excluded when we are computing the functional integral.

We will give the details of the ghost action later, but we note that the ghost field is also decomposed into transverse and longitudinal parts. This is given as

$$C_\mu = C_\mu^{\mathrm{T}} + \bar{\nabla}_\mu \eta, \tag{4.15}$$

$$\bar{C}_\mu = \bar{C}_\mu^{\mathrm{T}} + \bar{\nabla}_\mu \bar{\eta}, \tag{4.16}$$

$$b_\mu = b_\mu^{\mathrm{T}} + \bar{\nabla}_\mu \theta, \tag{4.17}$$

where C_μ^{T}, $\bar{C}_\mu^{\mathrm{T}}$, and b_μ^{T} are the transverse parts of the ghost field. Here we note that the lowest modes of the scalar fields η, $\bar{\eta}$, and θ are also not physical. Therefore we exclude these modes from the functional integral. We will also exclude the lowest modes of the fields C_μ and $\bar{C}_\mu$ since they satisfy the Killing equation, and are not true gauge degrees of freedom[2] [88].

Once these decompositions are implemented, in the flow equation we need a change of variable in the measure of the functional integral which will induce Jacobians. We can express these extra terms as an additional auxiliary action

$$\begin{aligned} S_{\mathrm{aux}} = &\int \mathrm{d}^D x \sqrt{\bar{g}} \left(\bar{\lambda} \mathcal{M}_\sigma \lambda + \omega \mathcal{M}_\sigma \omega + \bar{c}_\mu^{\mathrm{T}} \mathcal{M}_\xi^{\mu\nu} c_\nu^{\mathrm{T}} + \zeta_\mu^{\mathrm{T}} \mathcal{M}_\xi^{\mu\nu} \zeta_\nu^{\mathrm{T}} \right), \\ &+ \int \mathrm{d}^D x \sqrt{\bar{g}} \left(\bar{s} \mathcal{M}_\eta s + \bar{\psi} \theta \psi + w \mathcal{M}_\theta w \right), \end{aligned} \tag{4.18}$$

where we introduce the auxiliary fields. λ and $\bar{\lambda}$ are complex anti-commuting scalars, ω is a real commuting scalar, c_μ^{T} and $\bar{c}_\mu^{\mathrm{T}}$ are complex anti-commuting transverse vectors, ζ_μ^{T} is a real commuting transverse vector, s and $\bar{s}$ are complex commuting scalars, ψ and $\bar{\psi}$ are complex anti-commuting scalars, and finally ω is a real commuting scalar. The operators coming from the Jacobians are given by [10, 11]

$$\mathcal{M}_\sigma = \left[\left(1 - \frac{1}{D} \right) \bar{\nabla}^2 \bar{\nabla}^2 + \frac{\bar{R}}{D} \bar{\nabla}^2 \right]'', \tag{4.19}$$

$$\mathcal{M}_\xi = -2 \bar{g}^{\mu\nu} \left[\bar{\nabla}^2 + \frac{\bar{R}}{D} \right]', \tag{4.20}$$

$$\mathcal{M}_\eta = \mathcal{M}_\theta = \left[\bar{\nabla}^2 \right]'', \tag{4.21}$$

where the number of primes indicates the number of unphysical modes that have to be excluded from the functional integral. So in the FRGE equation, given in (2.29), we replace φ with a number of other operators. These are [10, 11]

$$\varphi = \{ h_{\mu\nu}^{\mathrm{T}}, \xi_\mu, \sigma, h, C_\mu^{\mathrm{T}}, \bar{C}_\nu^{\mathrm{T}}, \eta, \bar{\eta}, b_\mu^{\mathrm{T}}, \theta, \lambda, \bar{\lambda}, \omega, c_\mu^{\mathrm{T}}, \bar{c}_\mu^{\mathrm{T}}, \zeta_\mu^{\mathrm{T}}, s, \bar{s}, \psi, \bar{\psi}, w \}. \tag{4.22}$$

[2] The lowest modes of these ghost fields correspond to the lowest modes of the transverse vectors C^{T}, $\bar{C}_\mu^{\mathrm{T}}$ and b_μ^{T} and the second to lowest mode of the scalar parts of the ghost fields, η, $\bar{\eta}$ and θ.

So the overall contribution to the EAA becomes

$$\Gamma_k = -\frac{1}{16\pi G_k}\int \mathrm{d}^D x\sqrt{g}\,[R(g)-2\Lambda_k] + S_{\text{gauge}} + S_{\text{ghost}} + S_{\text{aux}}, \tag{4.23}$$

$$\begin{aligned} S_{\text{gauge}} = \frac{1}{2}\int \mathrm{d}^D x\sqrt{\bar g}\Bigg[\alpha\left(\bar\nabla^\sigma h_{\sigma\mu}\bar\nabla^\lambda h^\mu{}_\lambda - \left(\frac{1+\rho}{D}\right)^2 h\bar\nabla^2 h + \frac{2(1+\rho)}{D} h\bar\nabla_\mu\bar\nabla^\lambda h^\mu{}_\lambda\right) \\ +\beta\left(\bar\nabla^\sigma h_{\sigma\mu}\nabla^2\bar\nabla^\lambda h^\mu{}_\lambda - \left(\frac{1+\rho}{D}\right)^2 h\nabla_\mu\bar\nabla^2\nabla^\mu h + \frac{2(1+\rho)}{D}\right)\Bigg], \end{aligned} \tag{4.24}$$

$$S_{\text{ghost}} = \int \mathrm{d}^D x\sqrt{\bar g}\bar C_\mu \bar g^{\mu\lambda}(\alpha+\beta\bar\nabla^2)M^\nu_\lambda C_\nu + \frac{1}{2}\int \mathrm{d}^D x\sqrt{\bar g} b_\mu \bar g^{\mu\nu}(\alpha+\beta\bar\nabla^2)b_\nu. \tag{4.25}$$

We may now use this to compute the flow equation. There exists an extensive literature on the computation of the flow for the Einstein-Hilbert action [10, 81, 11, 14], most of which is beyond the scope of this thesis. The procedure itself is very technical. To give a qualitative insight into the procedure, this includes computing the Hessians[3] with respect to all the fields given in (4.22), and then finding the functional traces by using the Heat Kernel technique [114]. We wish to focus on the large-D results in this chapter. We note that in equations (4.24) and (4.25) we have three free parameters to fix. These are the gauge fixing parameters α, β, and ρ. We fix $\rho = 0$ and $\beta = 0$ for simplicity and retain only the α-dependence. We will talk about this dependence in due course. We also remind the reader that the flow equations given below are for the optimised cut-off function $R_{\text{opt}} = (k^2-q^2)\theta(k^2-q^2)$. Then the β-functions are [14]

$$\beta_\lambda = (-2+\eta)\lambda + g(a_1(\lambda) - \eta\, a_2(\lambda)), \tag{4.26}$$

$$\beta_g = (D-2+\eta)\, g, \tag{4.27}$$

$$\eta = \frac{g\, b_1(\lambda)}{1+g\, b_2(\lambda)}\,. \tag{4.28}$$

Here η is the anomalous dimension of the gravitational coupling constant and note that we rescale the gravitational coupling as $g \to \frac{g}{c_D}$ for simplification, where $c_D \equiv (4\pi)^{\frac{D}{2}-1}\Gamma(\frac{D}{2}+2)$. Coefficient functions, $a_1(\lambda)$, $a_2(\lambda)$, $b_1(\lambda)$, and $b_2(\lambda)$, are given in the appendix B.1.

We have the β-functions in terms of the total number of space-time dimensions D. Analytic equations for the non-trivial fixed points can be found by setting $\beta_\lambda = 0$ and $\beta_g = 0$ in equations (4.26) and (4.27), respectively. We find two equations,

[3] The Hessians are the $\Gamma_k^{(2)}$ terms that appear on the RHS of the FRGE in equation (2.29).

F_1 and F_2, for the fixed point of the gravitational coupling g from each of these differential equations as a function of the cosmological constant λ.

$$\begin{aligned} F_1 &= \frac{2-D}{b_1(\lambda)+(D-2)b_2(\lambda)}, \\ F_2 &= \frac{D\lambda}{a_1(\lambda)+(D-2)a_2(\lambda)}, \end{aligned} \tag{4.29}$$

hence $F_1 = F_2$ gives an analytical equation for λ. The equations we get are seventh order polynomials in λ and it is impossible to track the analytical solutions without fixing the dimension D and the gauge fixing parameter α. However, by looking at the $1/D$ expansion, one can find the leading order behaviour of the non-trivial fixed points in a large number of dimensions.

4.2 $1/D$ expansion of quantum gravity

4.2.1 Fixed points

We begin with a discussion of the fixed point solutions in the limit of a large number of dimensions. For this part of the analysis, we use the β-functions computed by the conventional background field techniques [14]. We have to distinguish between the cases where $0 \le \alpha < 1$, $\alpha = 1$ and $\alpha = \infty$. The reason for this is the following. The β-functions contain contributions from the cosmological constant, which acts as an "effective" mass term in the flow equation, with a mass (-2λ). It is a direct consequence of the IR regularisation that the flow equation has a putative pole in $(1-2\lambda)$, hence the bound $\lambda < \frac{1}{2}$. Due to the gauge fixing, the beta functions have an additional α-dependent contribution with effective mass term $(-2\alpha\lambda)$. The corresponding flow has a putative pole in $(1-2\alpha\lambda)$. For large D, the UV fixed point of λ approaches the boundary,

$$\lambda_{\text{bound}} = \frac{1}{2} \qquad (\text{or } \lambda_{\text{bound}} = \frac{1}{2\alpha} \text{ for } \alpha > 1). \tag{4.30}$$

For any $\alpha < 1$, the fixed point is determined by the α-independent terms in the β-functions, simply because $\lim_{\lambda\to 1/2}(1-2\lambda)/(1-2\alpha\lambda) = 0$ for any $\alpha < 1$. However, this limit is achieved more rapidly for smaller α. Furthermore, for $\alpha = 1$, the gauge-fixing begins to contribute at the same order as the α-independent terms. Therefore, the cases $\alpha < 1$ and $\alpha = 1$ are qualitatively different. Once $\alpha > 1$, the main contribution to the fixed point comes from the α-dependent terms. They dominate completely in the limit $\alpha \to \infty$, where even the fixed points begin to scale with $1/\alpha$. From this discussion, it is clear that the qualitative structure of the fixed point solutions change significantly depending on whether α is larger or smaller than one. It will be interesting to see this dependence explicitly for both the fixed points and universal observables.

We begin with an asymptotic expansion for the fixed points of λ and g, for the case $0 \leq \alpha < 1$. Inserting a polynomial ansatz in $1/D$ for λ into the fixed point condition leads to the series for the cosmological constant

$$\lambda(D,\alpha) = \frac{1}{2} - \frac{6}{D} + \frac{90}{D^2} - \frac{546 - 690\alpha}{(1-\alpha)D^3} - \frac{18(399 - 374\alpha + 167\alpha^2)}{(1-\alpha)^2 D^4} + \cdots . \tag{4.31}$$

Note that we do not denote the fixed points by an asterix $(*)$ here. We drop this notation for the remainder of this chapter.

Since the physical solution is restricted by (4.30), the expansion (4.31) describes the physical solution only for $\alpha < 1$. From the explicit result, we deduce that the effective expansion parameter in (4.31) is $[(1-\alpha)D]^{-1}$. Clearly, the limit $\alpha = 1$ cannot be achieved within this expansion. The rate of convergence is α-dependent. But also the first three non-trivial terms in the expansion (4.31) are α-independent.

Inserting the result (4.31) into F_1 or F_2, given in equation (4.29), leads to the corresponding expansion for the gravitational coupling. We find

$$g = \frac{6c_D}{D^3}\left(1 - \frac{26}{D} + \frac{425 - 473\alpha}{(1-\alpha)D^2} - \frac{5422\alpha^2 - 9500\alpha + 3214}{(1-\alpha)^2 D^3} + \cdots\right) . \tag{4.32}$$

Here, the first two non-trivial orders in the expansion are α-independent. Note that, c_D has been inserted back in. In marked contrast to the cosmological constant in (4.31), the gravitational coupling displays a non-trivial overall non-polynomial dependence on D. The leading order behaviour is

$$g = \frac{6}{D^3}(4\pi)^{D/2-1}\Gamma(\tfrac{D}{2}+2) \;+\; \text{subleading}\,, \tag{4.33}$$

which grows faster than the Γ-function.

For reasons given above, the expansions (4.31) and (4.32) have no smooth limit as $\alpha \to 1$. An explicit and direct analysis of the β-functions for $\alpha = 1$ gives the

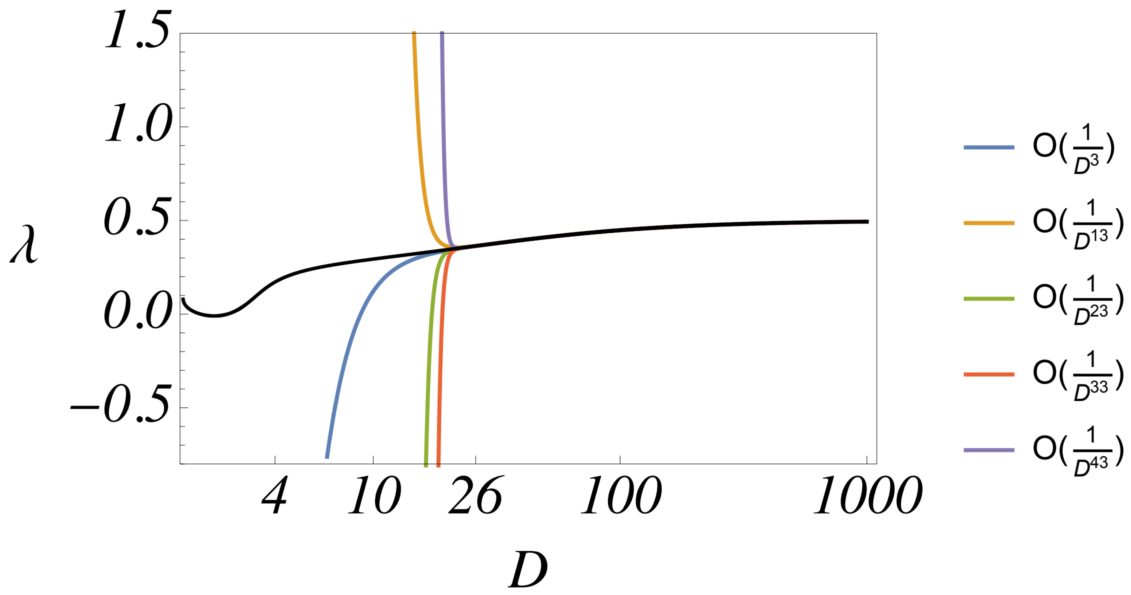

Figure 4.1: Variation of the fixed point of λ with D for $\alpha = 1$. The black curve represents the exact result and coloured curves represent different orders in $1/D$.

expansions

$$\lambda = \frac{1}{2} - \frac{6}{D} + \frac{90}{D^2} - \frac{678}{D^3} - \frac{2778}{D^4} + \cdots \tag{4.34}$$

$$g = \frac{6c_D}{D^3}\left(1 - \frac{28}{D} + \frac{525}{D^2} - \frac{6458}{D^3} + \frac{48739}{D^4} + \cdots\right) \tag{4.35}$$

The exact curve in figure 4.1 is computed analytically by solving $F_1 = F_2$ in equation (4.29). For large D, the exact fixed point as given in figure 4.1 is very well approximated by (4.34). The exact result is indistinguishable from the series (4.34) to order D^{-5} down to $D \approx 15$. The same holds true for $g^*(D)$. For large D, $g^*(D)$ is very well approximated by a $1/D$ expansion. Retaining the first six terms in (4.35), the result is barely distinguishable from the exact one down to $D \approx 25$ as can be seen in figure 4.2.

The radii of convergence of both of the expansions (4.31) and (4.34), computed using the equation (3.82), are found to be finite. We have computed these series to a high order in $1/D$. As a result, we found that the radius of convergence is constrained by $D \approx D_{\rm up}$, where, later in the chapter, $D_{\rm up}$ also denotes the dimension where the eigenvalues of the stability matrix at criticality become complex (bifurcation point). For (4.34), this means $D_{\rm up} \approx 25$. At $D = D_{\rm up}$, the qualitative picture changes. If the present picture – which predicts real critical exponents above $D_{\rm up}$ and complex ones below – is confirmed to higher order in the truncation, our observation suggests

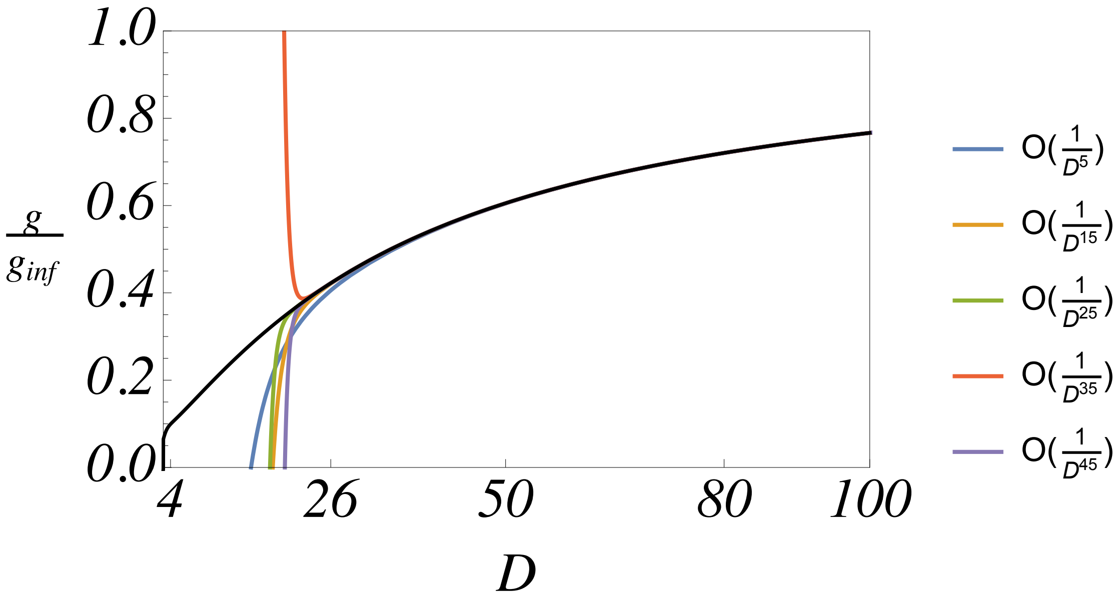

Figure 4.2: Variation of the fixed point of g with D for $\alpha = 1$. The black curve represents the exact result and coloured curves represent different orders in $1/D$.

that the physics for $D < D_{\rm up}$ is not well described within an expansion in $1/D$.

In summary, it is remarkable that the first three non-trivial orders of both (4.31) and (4.34) agree. For the gravitational coupling, we find that the leading order terms of both (4.32) and (4.35) agree. We conclude, that the non-trivial fixed points,

$$\lambda(D, \alpha \leq 1) = \frac{1}{2} - \frac{6}{D} + \frac{90}{D^2} \ + \ \text{subleading} \tag{4.36}$$

$$g(D, \alpha \leq 1) = \frac{6c_D}{D^3} \ + \ \text{subleading} \tag{4.37}$$

are gauge fixing independent.

4.2.2 Scaling exponents

Based on the results from the previous section, we can compute explicitly the eigenvalues at criticality for the various cases. For convenience, we introduce the notation

$$\left.\frac{\partial \beta_x(x,y)}{\partial y}\right|_{x_*,\, y_*} \equiv \beta_{x,y}$$

for the elements of the stability matrix with x and y given by λ or g.

With this, the elements of the stability matrix for $0 \leq \alpha < 1$ are given by

$$\beta_{\lambda,\lambda} = -\frac{D^3}{156}\left(1 + \frac{1}{13D} + \frac{16(3857 - 3350\alpha)}{169(1-\alpha)D^2} + \cdots\right), \tag{4.38}$$

$$\beta_{\lambda,g} = \frac{D^4}{156c_D}\left(1 + \frac{950}{13D} + \frac{107615\alpha - 115727}{169(1-\alpha)D^2} + \cdots\right), \tag{4.39}$$

$$\beta_{g,\lambda} = -\frac{2c_D}{13}\left(1 + \frac{12}{13D} + \frac{28068}{169D^2} - \frac{36(107389 - 166106\alpha + 111445\alpha^2)}{2197(1-\alpha)^2D^3} + \cdots\right), \tag{4.40}$$

$$\beta_{g,g} = -\frac{24D}{13}\left(1 - \frac{64}{13D} + \frac{7736}{169D^2} - \frac{32(61622 - 66185\alpha)}{2197(1-\alpha)D^3} + \cdots\right). \tag{4.41}$$

The first two (three) non-trivial orders of the matrix elements $\beta_{\lambda,\lambda}, \beta_{\lambda,g}$ ($\beta_{g,\lambda}, \beta_{g,g}$) are α-independent. The scaling exponents at the criticality with the gauge fixing constant at $\alpha < 1$ are found as

$$\theta_1 = \frac{D^3}{156}\left(1 + \frac{1}{13D} + \frac{8(6193\alpha - 7207)}{169(\alpha-1)D^2} + \cdots\right), \tag{4.42}$$

$$\theta_2 = 2D\left(1 + \frac{1}{D} + \frac{98}{D^2} - \frac{(920 - 1016\alpha)}{2(\alpha-1)D^3} + \cdots\right). \tag{4.43}$$

Note that off-diagonal elements coming from the coupling of cosmological and gravitational constants (i.e. mixing of $\sqrt{g}$ and $\sqrt{g}R$ terms in the Einstein-Hilbert action) affect the eigenvalues. When they are multiplied, the c_D dependence drops out. The leading order behaviour of the scaling exponent coming from the gravitational constant is $2D$, whereas the one from the cosmological constant is $\frac{D^3}{156}$. A high order in D means that λ approaches its fixed point very fast and is therefore not desirable.

The eigenvalues obey the relation

$$\frac{\theta_1}{\theta_2} = \frac{D^2}{312} + \text{ subleading}. \tag{4.44}$$

The bifurcation point is defined by $\theta_1 = \theta_2$. Extrapolating the large-D result down to finite D and solving for $\theta_1 = \theta_2$ gives

$$D_{\text{up,approx}} \approx 18. \tag{4.45}$$

Behaviour of the scaling exponents in a large number of dimensions can be seen in figure 4.3. This crude large-D estimate happens to be quite close to the value obtained without an expansion,

$$D_{\text{up}} \approx 25. \tag{4.46}$$

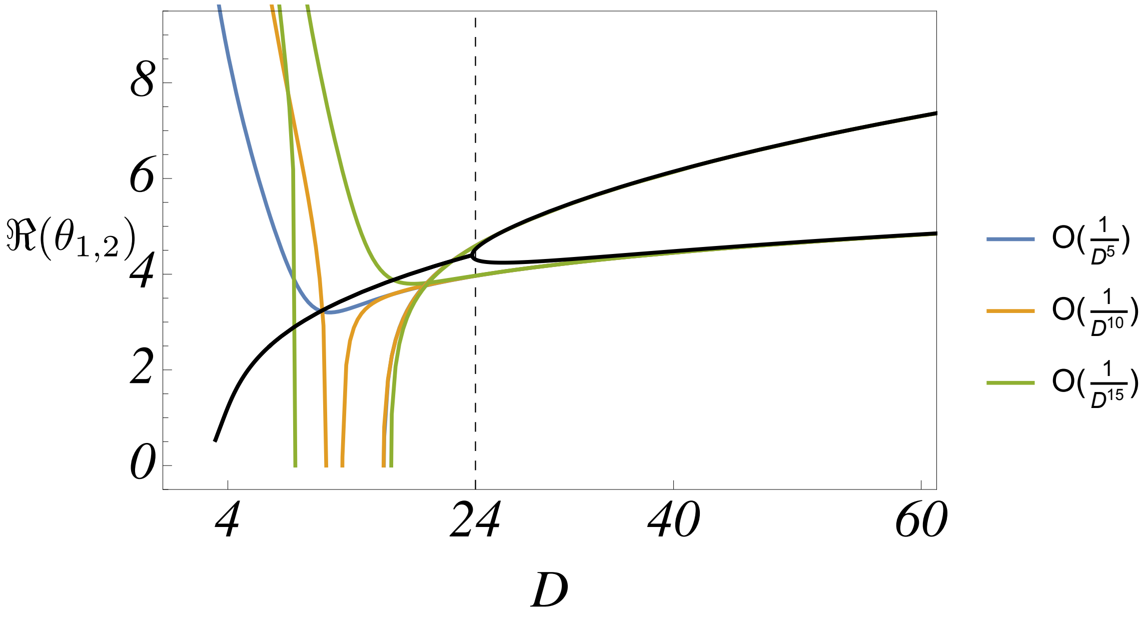

Figure 4.3: Variation of the scaling exponents with D where $\alpha = 0$. Black curves are the modulus of the exact value of the scaling exponents. Bifurcation from complex conjugate values to real values happens near $D = 24$ shown with the dashed line.

For $\alpha = 1$ we find

$$\theta_1 = \frac{D^3}{156}\left(1 + \frac{1}{13D} + \frac{50506}{169D^2} + \cdots\right), \tag{4.47}$$

$$\theta_2 = 2D\left(1 + \frac{1}{D} + \frac{98}{D^2} - \frac{37104}{13D^3} + \cdots\right). \tag{4.48}$$

Like the fixed points, α independent terms in the eigenvalues (4.42) and (4.43) are still the same and singularities are avoided. Therefore we conclude that the scaling exponents are gauge independent on the leading terms as well.

4.3 Approximations

4.3.1 Vector dominance gauge

For the approximation where we take the asymptotic limit for $1/\alpha = 0$, in the coefficient functions, the terms $(1-2\alpha\lambda)^{-1}$ dominate. These terms are coming from the vector traces in the flow equation, therefore we call it the vector dominance gauge. In the appendix, (B.7) and (B.8), the full explicit solution for $\lambda(D,\infty)$ and $g(D,\infty)$ is given. For this case they are rescaled as $\tilde{\lambda} = \frac{\lambda}{\alpha}$ and $\tilde{g} = \frac{g}{\alpha}$.Expanding the explicit result around $D \to \infty$, we find

$$\tilde{\lambda}(D,\infty) = \frac{1}{2} - \frac{1}{(2D)^{1/2}} + \frac{1}{D} - \frac{5}{(2D)^{3/2}} + \frac{3}{D^2} - \frac{75}{2(2D)^{5/2}} + \frac{13}{D^3} - \cdots , \quad (4.49)$$

$$\tilde{g}(D,\infty) = \frac{c_D}{2D^2}\left(1 - \frac{4}{(2D)^{1/2}} + \frac{7}{D} - \frac{44}{(2D)^{3/2}} + \frac{37}{D^2} - \frac{478}{(2D)^{5/2}} + \cdots\right). \quad (4.50)$$

We note that the effective expansion parameter is $1/\sqrt{D}$. This is different from the $0 \leq \alpha \leq 1$ cases. When $\alpha > 1$, the coupling constants significantly scale with α. We see that λ still has the same leading order term ($\frac{1}{2}$ in large-D) and the leading term in g is still proportional to the factor c_D.

Scaling exponents

The scaling exponents for the $\alpha \to \infty$ case are given by

$$\theta_1 = \sqrt{2}D^{3/2}\left(1 - \frac{13}{2D} - \frac{4\sqrt{2}}{D^{3/2}} - \frac{9}{8D^2} - \cdots\right), \quad (4.51)$$

$$\theta_2 = 2D\left(1 + \frac{2}{D} + \frac{4\sqrt{2}}{D^{3/2}} + \frac{22}{D^2} + \cdots\right). \quad (4.52)$$

We can easily see that θ_2 has the same first order term $2D$, whereas the leading term in θ_1 is pulled down to $D^{3/2}$. The scaling exponent coming from the cosmological constant gives a more reasonable behaviour as the leading power of D is lower. This also means that θ_1 has a gauge dependence for this extreme case. This is due to the fact that the terms coming from the vector traces in the flow equation dominate in this extreme limit.

4.3.2 Bimetric Truncation

A recent study showed a different approach to background independence of quantum gravity [113]. In this approach the gravity part of the action is separated into two parts, a background and a dynamical part so that two sets of β-functions are obtained. This is known as the bimetric truncation. The gravitational action ansatz is written as

$$\begin{aligned}\Gamma_k = &- \frac{1}{16\pi G_k}\int \mathrm{d}^{\mathrm{d}}x\sqrt{g}\,(R - 2\Lambda_k) \\ &- \frac{1}{16\pi G_k^B}\int \mathrm{d}^{\mathrm{d}}x\sqrt{\bar{g}}\left(\bar{R} - 2\Lambda_k^B\right),\end{aligned} \quad (4.53)$$

where a bar on a symbol represents background terms and a superscript B, represents the couplings of the background field.

The aim of this truncation is to track down the background independence and check whether asymptotic safety and background independence can exist together. In [113] a new method was developed to compute the RG equations. A conformal projection is used and in the beginning of the calculation the gauge is fixed as $\alpha = 1$. This gauge fixing is justified as it leads to simplifications in the calculation of the flow.

By using the D dependent flow equations from coefficient functions given in equations (B.9)- (B.12), we find the $1/D$ expansion of the fixed points for the dynamical part of the action[4] as

$$\lambda = -\frac{6}{D} - \frac{42}{D^2} + \frac{684}{D^3} + \frac{2256}{D^4} + \cdots , \tag{4.54}$$

$$g = \frac{6c_D}{D^3}\left(1 + \frac{27}{D} + \frac{193}{D^2} - \frac{2321}{D^3} + \frac{3165}{D^4} + \cdots\right). \tag{4.55}$$

We observe that the leading order term in g is $6c_D/D^3$ just as in the conventional case, whereas the leading term in λ, which was $\frac{1}{2}$, disappears in the bimetric truncation. Furthermore, λ approaches zero from the negative domain in the large-D limit. This can be seen in figure 4.4.

The reason for this behaviour lies in the β-functions. To see this we perform a simple analysis of the coefficient functions. In the single metric case, λ approaches $1/2$ as the term $(1-2\lambda)$ approaches its effective value $-6/D$, i.e. the second term in the expansion. In order to get a fixed point in the large-D, $b_1(\lambda)$ has to have a positive sign. We cannot get the right sign unless we have an expansion for λ with $1/2$ as the leading order term. When this is satisfied, the leading two terms coming from the tensor and scalar traces cancel each other and the term coming from the spin-1 trace gives the fixed point. In the bimetric truncation, we do not need this. For the bimetric truncation $b_1(\lambda)$ already has the right sign and λ is allowed go below $1/2$ and indeed to zero.

[4] We are not able to give expressions for the background couplings as the original paper, [113], includes some typos in the expressions for the D-dependent β-functions. Although we note that the dynamical β-functions only depend on the dynamical coupling constants, whereas background β-functions depend on both the dynamical and the background coupling constants.

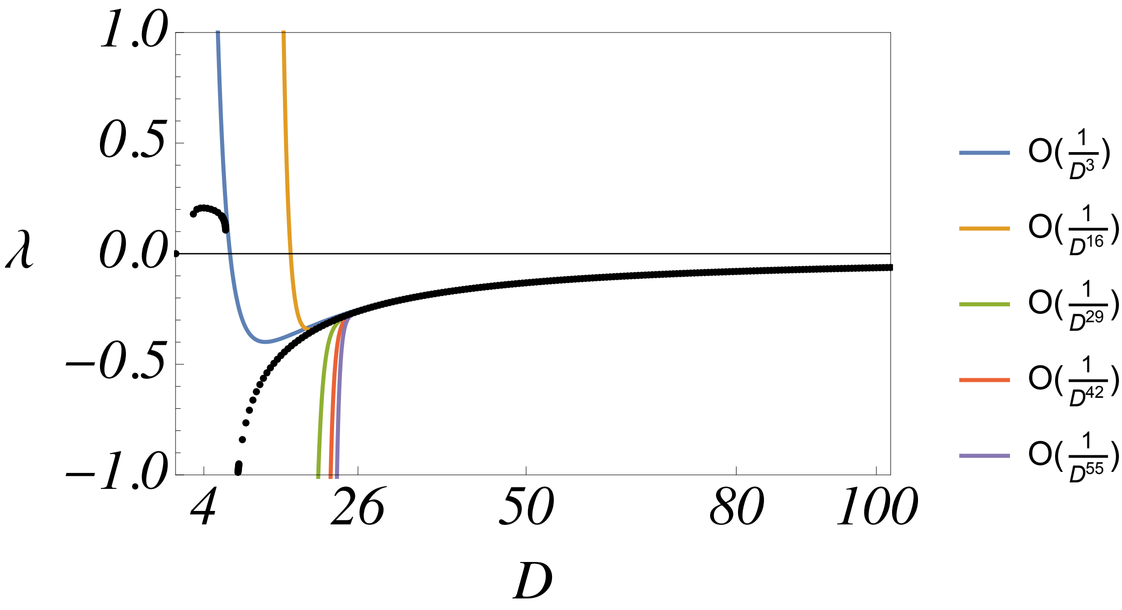

Figure 4.4: Variation of the fixed point of λ with D in the bimetric truncation. The black curve represents the exact result and coloured curves represent different orders in $1/D$.

We also find there are no real fixed points between dimensions $D \approx 7$ and $D \approx 8$, where the fixed point jumps to the complex plane. Around $D \approx 9$ a second fixed point with positive scaling exponents emerges for a short time and then disappears around $D \approx 10$. This second fixed point is not shown in the figures. The role of this second fixed point may be the topic of a further study as it may have physical implications. However, this falls beyond the scope of our study as we are interested to follow the other fixed point which is investigated in four dimensions in [113]. After this second fixed point disappears, the fixed point for the cosmological constant approaches zero from the negative plane. This explains the behaviour of the black curves in low dimensions in figures 4.4 and 4.5.

We find the radius of convergence for this expansion to be around $D \approx 25$ as similar to the single-metric case. This value, both can be seen in figures 4.4 and 4.5 and also can be computed by using equation (3.82).

Scaling exponents

The elements and the eigenvalues of the stability matrix for the dynamical part of the bimetric action are found in a similar way to the single metric case. We find the

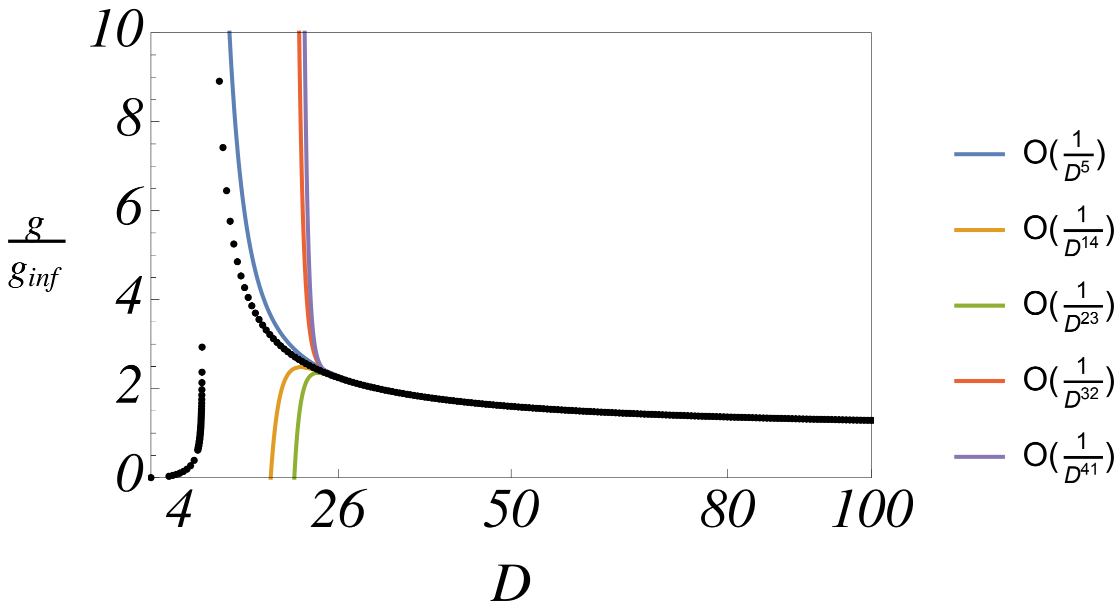

Figure 4.5: Variation of the fixed point of g with D in the bimetric truncation. The black curve represents the exact result and coloured curves represent different orders in $1/D$.

elements of the stability matrix as

$$\beta_{g,g} = -2D - 8 - \frac{96}{D} + \frac{880}{D^2} + \frac{216}{D^3} + \cdots , \tag{4.56}$$

$$\beta_{g,\lambda} = -\frac{24}{D^2} + \frac{144}{D^3} + \frac{2304}{D^4} - \frac{70560}{D^5} + \cdots , \tag{4.57}$$

$$\beta_{\lambda,g} = -3D^2 + 87D - 1626 + \frac{16220}{D} + \cdots , \tag{4.58}$$

$$\beta_{\lambda,\lambda} = -D - \frac{180}{D} + \frac{3336}{D^2} - \frac{432}{D^3} + \cdots . \tag{4.59}$$

Eigenvalues of this matrix give the scaling exponents as the following

$$\theta_1 = D + \frac{108}{D} - \frac{1392}{D^2} - \frac{29520}{D^3} + \cdots , \tag{4.60}$$

$$\theta_2 = 2D - 8 + \frac{162}{D} - \frac{2824}{D^2} + \frac{29736}{D^3} + \cdots . \tag{4.61}$$

Note that the leading order term of the exponent which comes from the gravitational constant stays the same as in all the cases we investigated previously in this chapter. The eigenvalue that comes from the cosmological constant shows a significantly more stable behaviour with a lower power of D. We also note that the scaling exponents are both positive and real, whereas in four dimensions they are

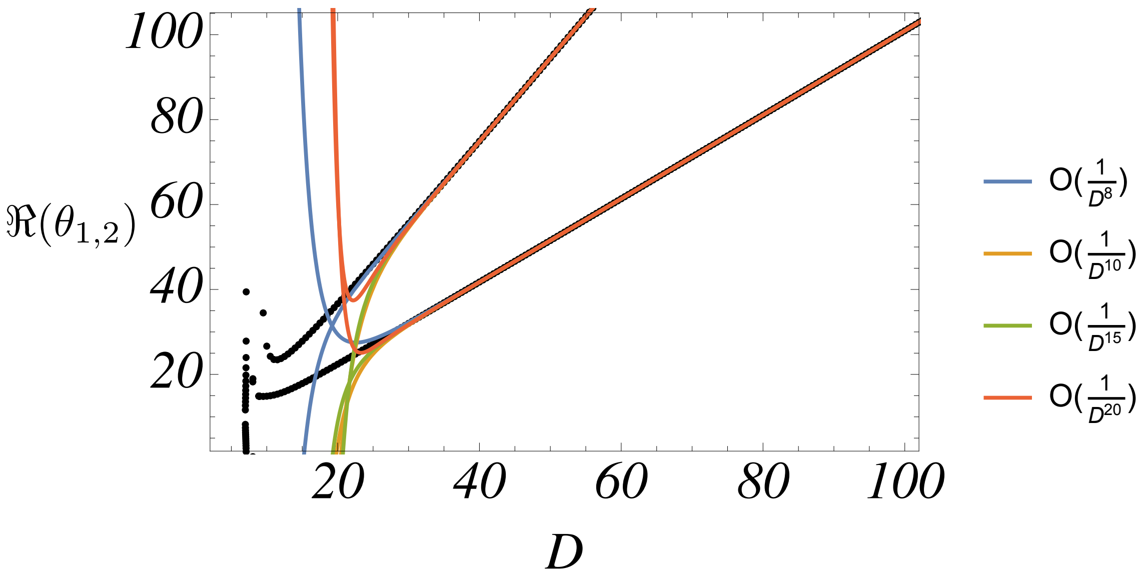

Figure 4.6: Variation of the scaling exponents with D in the bimetric truncation. The black curve represents the exact result and coloured curves represent different orders in $1/D$.

the complex conjugate of each other. Looking at the leading order terms we can expect to see a bifurcation at $D = 8$ and, in fact, from the exact equation we find the bifurcation point is found to be around $D \approx 7$. The bifurcation point cannot be seen as clearly as in the single metric case due to the fact that the values jump in and out of the domain where the scaling exponents are either positive or complex conjugate of each other. This can be observed in figure 4.6.

4.3.3 Static cosmological constant

In this section we analyse the fixed points for a 'static' cosmological constant, meaning that the non-trivial running for λ is neglected and its running is replaced by the canonical one, $\bar{\lambda}_k = \lambda\, k^2$. In this approximation, a constant dimensionful cosmological constant $\bar{\lambda}$ corresponds to a scaling $\lambda \sim 1/k^2$ for the dimensionless λ. We only look into results computed using the background field technique. A very similar analysis can be (and has been) carried out for the bimetric truncation. Qualitatively similar results can be drawn for the bimetric truncation.

The renormalisation group equation for the gravitational coupling at when $\lambda =$

constant is given by

$$\beta_g = (D - 2 + \eta)\, g\,, \quad \eta = \frac{g\, b_1(\lambda)}{1 + g\, b_2(\lambda)}\,. \tag{4.62}$$

The anomalous dimension is then evaluated at an appropriate value for λ. The β-function in (4.62) gives a Gaussian fixed point at $g^* = 0$ and a non-trivial fixed point at

$$g^* = \frac{D-2}{(2-D)b_2 - b_1}\,. \tag{4.63}$$

Interestingly, in the large-D limit and when the gauge fixing constant is $0 \leq \alpha \leq 1$, we find

$$g^* = -\frac{6c_D}{D^3} + \text{subleading}. \tag{4.64}$$

In other words, the gravitational interactions become repulsive in the high energy limit without the running of the cosmological constant.

We find the scaling exponent by using the relation,

$$\left.\frac{\partial \beta_g}{\partial g}\right|_* = -\theta\,. \tag{4.65}$$

Therefore

$$\theta_{\rm NG} = (D-2) + \frac{b_2}{b_1}(D-2)^2 \tag{4.66}$$

at the non-Gaussian (NG) ultraviolet fixed point (4.63), and

$$\theta_{\rm G} = 2 - D \tag{4.67}$$

at the Gaussian (G) fixed point. Hence, the explicit solution to the flow equation (4.62) for arbitrary scales k is given by

$$\frac{k}{\Lambda} = \left|\frac{g_k}{g_\Lambda}\right|^{-1/\theta_{\rm G}} \left|\frac{g^* - g_k}{g^* - g_\Lambda}\right|^{-1/\theta_{\rm NG}}\,, \tag{4.68}$$

so long as the initial condition g_Λ is different from the fixed point values. The expressions in equation (4.62) are verified by differentiating the solution given in (4.68) with respect to $t = \ln k$ and making use of equations (4.63), (4.66), and (4.67). The explicit solution is valid for an arbitrary cut-off function and any fixed value for the dimensionless cosmological constant. The universal eigenvalue (4.66) depends weakly on these parameters.

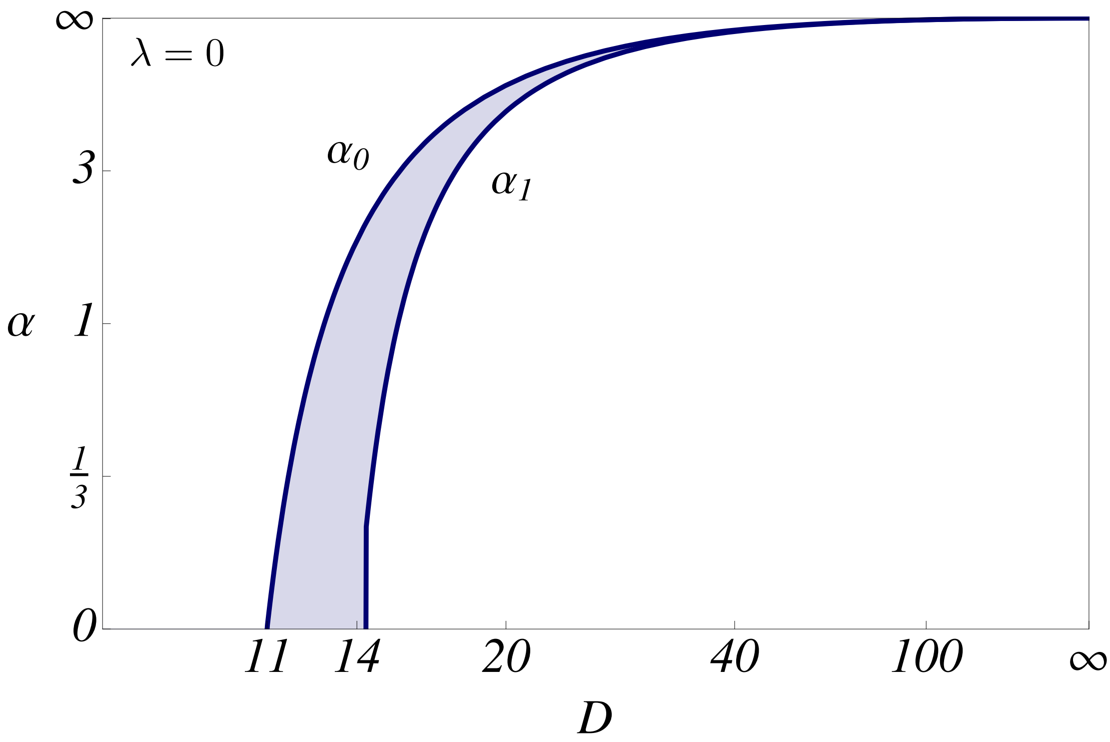

Figure 4.7: Domain of validity for $\theta_{\rm NG}$ with a vanishing cosmological constant for various gauge choices. On the right curve, $\theta_{\rm NG}$ diverges and on the left curve $\theta_{\rm NG} = 0$. Hence, in between the lines $\theta_{\rm NG}$ has a negative value which is the wrong sign for the scaling exponent. Outside the blue area is the domain of validity, which increases with increasing dimensions.

It is interesting to discuss the numerical values of equation (4.66) as a function the gauge fixing parameter. We can explicitly write the expression for the eigenvalue of the stability matrix as

$$\theta_{\rm NG} = 2\,\frac{D-2}{D+2}\,\frac{D^6 - 13D^5 + D^4(31-24\alpha) + 3D^3(24\alpha-35) + D^2(74-48\alpha) + 36D + 24}{D^5 - 16D^4 + D^3(39-24\alpha) + 36D^2(2\alpha-3) + D(84-48\alpha) + 24}. \tag{4.69}$$

It can be seen in equation (4.69) that the leading term is $2D$.

For a finite α and in the limit $D \to \infty$, we find

$$\theta_{\rm NG} = 2D\,\frac{D-2}{D+2}\,\frac{D-13}{D-16} \tag{4.70}$$

up to correction of order $1/D^2$ and α/D^3. Hence, equation (4.70) indicates that a $1/D$ expansion has a finite radius of convergence given by $D \approx 16$, while an

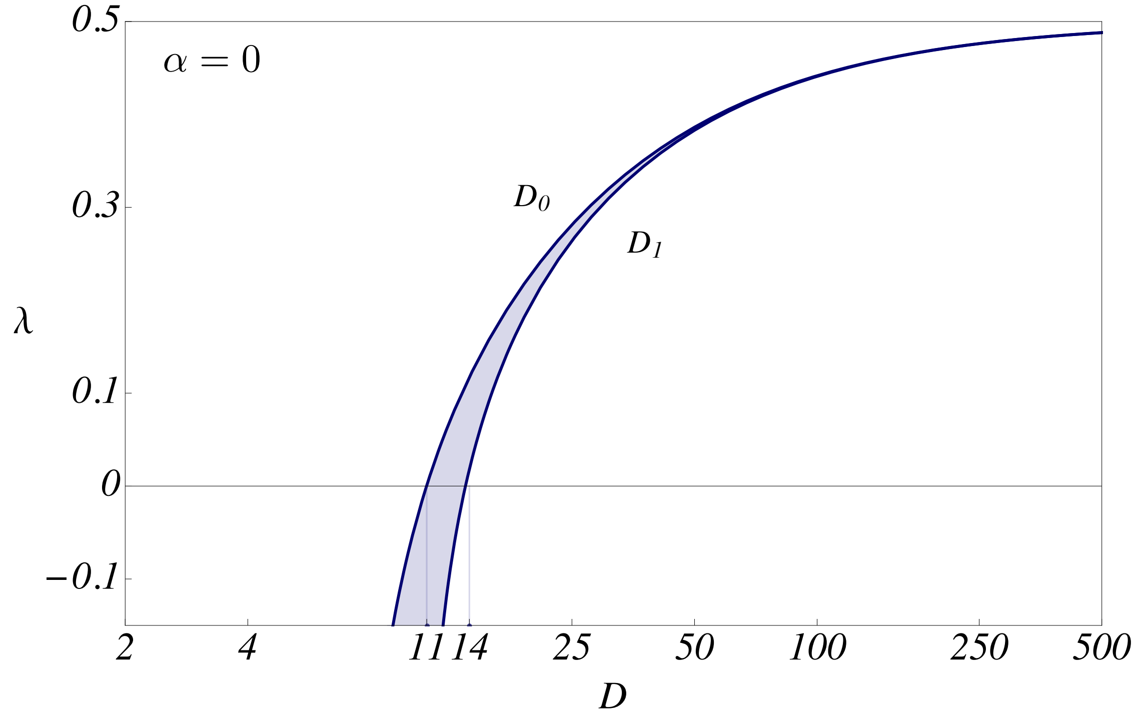

Figure 4.8: Domain of validity for θ_{NG} with a constant cosmological constant at $\alpha = 0$. A positive value for the cosmological constant makes the area in between the curves smaller and smaller increasing the domain of validity as $\lambda \to 1/2$ with increasing D.

expansion in D is limited by $D \approx 13$. A more detailed inspection of equation (4.69) shows that $\theta_{\rm NG}$ vanishes at

$$\alpha_0 = \frac{D^6 - 13D^5 + 31D^4 - 105D^3 + 74D^2 + 36D + 24}{24D^2(D-2)(D-1)}. \tag{4.71}$$

Solving the equation (4.71) for D gives a function that we name $D_0(\alpha)$. Assuming $D > 2$, this leads to a monotonically increasing function which starts at $D_0 = 10.99$ for $\alpha = 0$. The vanishing of $\theta_{\rm NG}$ indicates a qualitative change in the solution and marks the domain of validity of the approximation. Furthermore, $\theta_{\rm NG}$ diverges at

$$\alpha_1 = \frac{D^5 - 16D^4 + 39D^3 - 108D^2 + 84D + 24}{24D(D^2 - 3D + 2)}\,. \tag{4.72}$$

For all $\alpha \geq 0$ and $D > 2$, we observe that $1/\theta_{\rm NG}$ vanishes at $D_1(\alpha) > 0$, see figure 4.7. The function $D_1(\alpha)$ starts at $D_1 = 13.69$ for $\alpha = 0$. Hence, the monotonically increasing function $D_1(\alpha)$ provides a limit for an expansion in $1/D$. We note that $D_0(\alpha) \leq D_1(\alpha)$, while equality is reached for $1/\alpha = 0$ and $1/D_0 = 1/D_1 = 0$. It is interesting to note that a positive cosmological constant $\lambda > 0$, set to zero in (4.69), has a stabilising effect. For both cases, with increasing $\lambda \in [0, \frac{1}{2}]$, the boundaries D_0 and D_1 are pushed to larger values, increasing the domain of validity of the approximation. This is shown in figure 4.8 for $\alpha = 0$. A similar pattern holds for all α. In particular, for $\lambda \to \frac{1}{2}$ and $\alpha \in [0, 1]$ (or $\lambda \to 1/2\alpha$ for $\alpha \in [1, \infty]$), we have

$$\theta_{\rm NG} = 2D\frac{D-2}{D+2}\,. \tag{4.73}$$

Hence, the scaling exponent $\theta_{\rm NG}$ stays positive and finite for all $D > 2$.

Decoupling of the cosmological constant

In light of these results, we finally consider the limit $1/\alpha \to 0$. From the structure of the β-functions it is evident that a potential fixed point for the cosmological constant should scale as $1/\alpha$. Therefore, the approximation $\lambda = 0$ correctly takes the non-trivial λ fixed point into account, which accidentally happens to be degenerate with the Gaussian fixed point. For $\alpha \to \infty$ and arbitrary $\lambda \neq \frac{1}{2}$, we find

$$\theta_{\rm NG} = 2D\frac{D-2}{D+2} \tag{4.74}$$

up to subleading corrections in $1/D$ and in λ/D. Note that equation (4.74) is identical to the result in (4.73), yet achieved in a different limit. Clearly, (4.74) stays negative and finite for all $D > 2$, allowing for a smooth continuation between small and large numbers of dimensions. We note that the limits $1/D \to 0$ and $1/\alpha \to 0$ of equation (4.69) are not the same, as is evident from equations (4.70) and (4.74). Still, their leading order behaviour is identical. We conclude that equation (4.74) can be regarded as the more stable approximation in the limit of higher dimensions. A more detailed understanding of the large dimensional limit additionally necessitates the non-trivial running of the cosmological constant.

4.3.4 Structural stability of the fixed points

Finally, we look into the case where we ignore the quantum corrections to the anomalous dimension. In equations (4.28) and (4.26), the functions a_2 and b_2 emerge from the wave function renormalisation $Z_{N,k}$ and lead to a Hartree-Fock type resummation. In the vicinity of the fixed points, due to large anomalous dimension, these terms have a significant contribution. We compute the series expansion of the fixed points without these terms, in order to be able to compare the leading order behaviour in the large-D. The results can be seen in tables 4.1 and 4.2; table 4.1 for the leading order behaviour of the fixed points in the stated approximations and 4.2 is for the scaling exponents. In these tables, only the fixed points with positive gravitational constant are considered and positive scaling exponents are preferred over negative ones.

	Single Metric		Bimetric	
	λ	g	λ	g
$a_2 = 0, b_2 \neq 0$	$1/2$	$12/D^3$	-	-
$a_2 \neq 0, b_2 = 0$	$1/2$	$6/D^3$	$-12/D$	$12/D^3$
$a_2 = 0, b_2 = 0$	$1/2$	$12/D^3$	$-D^2/16$	$12/D^3$
$a_2 \neq 0, b_2 \neq 0$	$1/2$	$6/D^3$	$-6/D$	$6/D^3$

Table 4.1: Leading order terms in fixed points with different approximations.

Fixed points with the correct sign of scaling exponent exist in all approximations except for the $a_2 = 0$ case in the bimetric truncation. This can be explained by looking into the β-functions carefully. In the β-function of the cosmological constant,

	Single Metric		Bimetric	
	θ_1	θ_2	θ_1	θ_2
$a_2 = 0, b_2 \neq 0$	$D^3/42$	D	-	-
$a_2 \neq 0, b_2 = 0$	$D^3/576$	$2D$	D	D
$a_2 = 0, b_2 = 0$	$D^3/144$	$7D/12$	D	D
$a_2 \neq 0, b_2 \neq 0$	$D^3/156$	$2D$	D	$2D$

Table 4.2: Leading order terms in scaling exponents with different approximations.

a_2 is responsible for the existence of the fixed point in the first place. Thus, excluding a_2 causes the fixed point to disappear. Also, unlike the low dimension case, we get real scaling exponents in all cases which point to a bifurcation in scaling exponents as a function of D, just like the behaviour of the black curve in figure 4.3.

4.4 Naive dimension analysis

Naive dimension analysis (NDA), first introduced in [115], developed for QCD and supersymmetric theories [116], and later extended to gravity in extra dimensions [117]. In the case of gravity, it provides an estimate of how strong the gravitational interactions are. According to NDA, if each of the loops contributed to the β-function the same, this would give us a coupling $g_{\text{NDA}} = (4\pi)^{D/2}\Gamma(D/2)$ [117]. If the loops contribute less than this factor, each loop would be smaller than the previous one and the theory would be a perturbative theory, however if they contribute more than this factor then it means the effect of each additional loop is more than the previous one and the theory becomes unsolvable. Therefore we can conclude that g_{NDA} acts as a strong coupling limit.

Our results from different approximations give us a gravitational constant $g \approx 6c_D/D^3$, where $c_D = (4\pi)^{D/2-1}\Gamma(D/2+2)$. It seems like, in the large-D limit, g increases with the Γ-function. Comparing this result with the NDA result gives

$$\frac{6c_D/D^3}{g_{\text{NDA}}} = \frac{3(D+2)}{8\pi D^2}. \tag{4.75}$$

In this comparison, we see that the large-D result is smaller than the strong coupling limit. This naive analysis suggests that our results are not in danger of being too strongly coupled at high energies in large-D. Therefore the fixed point value in the large-D limit is well within the bounds of reliablilty.

4.5 Discussion

With the help of functional renormalisation group techniques, we are able to give a quantum gravity picture of the large-D expansion. From a single metric action, UV fixed points with the right properties exist in the large-D with a finite radius of convergence of $D \approx 25$. At the same time, $D \approx 24$ is where the eigenvalues of the stability matrix bifurcate. Therefore we can argue that the physical mechanism behind the bifurcation around this dimension, does not let the series expansion go further below, e.g. down to $D = 4$. Nonetheless we have a very clear picture of the dynamics of the theory at large-D.

We also looked into the case of bimetric truncation. It is very interesting to see that, similar to the single metric truncation, the series expansion has a finite radius of convergence around $D \approx 25$. However this time, bifurcation of the scaling exponents is $D \approx 7$ due to the fluctuations of the fixed points between $D \approx 7$ and $D \approx 10$.

In the single metric case, we have calculated the expansions up to very high orders in D with various gauge choices and shown that gauge dependence is only in the subleading terms in the expansion. The first two (three) leading order terms of the gravity (cosmological constant) coupling fixed point is gauge fixing independent. On the other hand when we look into an extreme case where the gauge fixing constant tends to infinity, we observe that the leading order power of the scaling exponent that comes from the cosmological constant decreases. Unfortunately, we cannot do the same analysis for the bimetric case due to the fixing of the gauge parameter from the beginning.

In the case where we assumed a zero cosmological constant, we have found a negative gravitational constant with the correct sign of scaling exponent, only to turn positive as soon as we switch on the cosmological constant.

It is important to note that the scaling exponent for the gravitational constant from different approximations gives a very stable leading order value of $2D$. This result is consistent with the large-D, lattice quantum gravity studied in [118].

Although we provide a fixed point analysis for the Einstein-Hilbert truncation, this work can be extended by using a more general quantum gravity action such as $f(R)$ gravity or a case where gravity couples to matter. Apart from this, our analysis can be extended to explain some of the results that we left unexplained. Firstly, a deeper investigation may point us towards an explanation as to why the radius of convergence of our theory is very close to the bifurcation point of the scaling exponents. The dimensions $D \approx 25$ may or may not correspond to the dimensions of bosonic string theory, where $D = 26$. Secondly, the appearance of the second fixed point in the bimetric approximation may be investigated further to give the underlying physical explanation. Finally, a large-D analysis may be done on various other approximations within asymptotically safe gravity such as [119] where we might expect to get similar results to our static cosmological constant case, or [120] where spectral sums used instead of heat kernels as in our case.

All in all this work may be considered as a significant leap forward in the area of large-D expansion of quantum gravity as we have successfully obtained a large-D expansion within the functional renormalisation group framework.

Chapter 5

Conclusion

We started by asking two questions and promised to address these questions by using asymptotic safey and a limit where the number of group dimensions is large. Here we explain how we addressed these questions and summarise our results. Before addressing the questions, in chapter 2 we had provided tools to investigate asymptotic safety of non-perturbatively renormalisable theories. We reviewed the formalism of Wilsonian renormalisation and derived the exact renormalisation group equation.

Question 1: What is beyond the standard model?

We address this question in chapter 3 by using a model in which we have a large number of particles. This thesis focuses on the effect of higher order scalar self-interactions on the asymptotic safety properties of four dimensional gauge-Yukawa theories. Here we established that the asymptotic safety can be realised with the inclusion of higher dimensional operators with only one relevant operator. In fact, even if we include an arbitrary number of scalar self-interactions of type ρ_1^n and $\rho_1^n\rho_2$, the asymptotic safety still exists as long as we are in the Veneziano limit where the expansion parameter is small, $0 < \epsilon \ll 1$. One of the key results of this chapter is that the numerical values of the higher order fixed points are iteratively higher order in ϵ. Since the value of ϵ is small, we say that the potential has perturbative properties since the higher order couplings contribute less than the previous order.

Furthermore we obtain a closed form expression for the coupling constant fixed point values. We showed analytically that the expressions, that we had found to leading order in ϵ, are resummable. Resummation of the potential with the higher order coplings brings a logarithmic contribution to the potential. We also show analytically that the scaling exponents are truly universal in that they are independent of the regulator choice. We show this up to $\mathcal{O}(\epsilon)$. This analytic understanding gives us stronger confidence in the result that the inclusion of higher dimensional scalar self-interactions does not ruin the asymptotic safety of the theory.

Question 2: What is the nature of gravity?

Although there are so many ways to take in this quest, in this thesis we are interested in asymptotic safety, since asymptotic safety relies on the robust tools of quantum field theory to investigate the interactions of gravity. Given the success of quantum field theory in many areas of physics, one can expect this to be a promising approach to gravity. Also, by definition, this enables us to question whether quantum gravity is a predictive and fundamental theory of nature. In chapter 4 we address this question by looking into a model with a large number of dimensions, D. For gravity, on top of being the group degrees of freedom of gravity's local gauge group, spacetime dimensions, comes from the loop factors.

We use the Einstein-Hilbert truncation where we only retain the cosmological constant and the Ricci scalar as operators. We use the β-functions that were scrutinised in [10, 81, 11, 14]. We give the β-functions as a function of D and keep the gauge fixing α-dependence explicit. We find the fixed points as an expansion in $1/D$, and we find that the expansion has a finite radius of convergence. Even though we cannot go down to dimension four, the expansion is valid down to dimensions $D \sim 25$. This also happens to be the bifurcation point of the scaling exponents where the scaling exponents turn from being a complex conjugate pair to two real numbers. We also look into various other approximations such as a vector dominance gauge [121], bimetric approach for the metric expansion [113], as well as a static cosmological constant. One thing in common in all these approaches is that the scaling exponent associated to the gravitational constant is always $2D$. This is consistent with results from lattice quantum gravity [118].

As a final remark, asymptotic safety is a very important property to look for in a fundamental theory of nature. The functional renormalisation group equips us with very useful tools to investigate asymptotic safety of perturbatively non-renormalisable theories such as the ones we have investigated here. We acknowledge that there is of course a long way to go in order to fully answer the questions we posed. This thesis offers a small step towards addressing some of the most important questions of nature.

Appendix A

Beyond marginal operators

A.1 Scalar Sector Mass Spectrum

We use the reference [65] as a guideline to write the scalar mass spectrum. We parameterise the field such that $h = \{h_{\rm ab}\}$, where $h_{\rm ab} = h_{\rm a}\delta_{\rm ab}$. Recall that h is a complex $N_f \times N_f$ matrix. The mass terms for our potential can be found as

$$\begin{aligned}\frac{\delta^2 v_k}{\delta h^{\rm R}_{\rm ab}\delta h^{\rm R}_{\rm cd}} =& v_k'\delta_{\rm ac}\delta_{\rm bd} + 2v_k'' h_{\rm a}h_{\rm c}\delta_{\rm ab}\delta_{\rm cd} + \frac{\partial v_k}{\partial\rho_2}\frac{\delta^2\rho_2}{\delta h^R_{\rm ab}\delta h^R_{\rm cd}} \\ &+\sqrt{2}\frac{\partial v_k'}{\partial\rho_2}\left(h_{\rm a}\delta_{\rm ab}\frac{\delta\rho_2}{\delta h^R_{\rm cd}} + h_{\rm c}\delta_{\rm cd}\frac{\delta\rho_2}{\delta h^R_{\rm ab}}\right), \end{aligned} \tag{A.1}$$

$$\frac{\delta^2 v_k}{\delta h^{\rm I}_{\rm ab}\delta h^{\rm I}_{\rm cd}} = v_k'\delta_{\rm ac}\delta_{\rm bd} + \frac{\partial v_k}{\partial\rho_2}\frac{\delta^2\rho_2}{\delta h^I_{\rm ab} h^I_{\rm cd}}, \tag{A.2}$$

$$\frac{\delta^2 v_k}{\delta h^{\rm I}_{\rm ab}\delta h^{\rm R}_{\rm cd}} = 0. \tag{A.3}$$

where $h_{\rm ab} = \frac{1}{\sqrt{2}}\left(h^R_{\rm ab} + i h^I_{\rm ab}\right)$ and prime means partial derivative with respect to ρ_1.

The mass spectrum can be calculated using these equations by finding the eigenvalues. These are provided in reference [65].

Appendix B

1/D

B.1 The flow from single metric

In the single metric approximation, for an arbitrary gauge fixing parameter α, the coefficient functions in equations (4.26)-(4.28) are given by

$$a_1(\lambda) = \frac{D(D-1)(D+2)}{2(1-2\lambda)} + \frac{D(D+2)}{1-2\alpha\lambda} - 2D(D+2)\,, \tag{B.1}$$

$$a_2(\lambda) = \frac{D(D-1)}{2(1-2\lambda)} + \frac{D}{1-2\alpha\lambda}, \tag{B.2}$$

$$\begin{aligned} b_1(\lambda) = & -\tfrac{1}{3}(1+\tfrac{2}{D})(D^3+6D+12) + \frac{D(D+2)(D^3-2D^2-11D-12)}{12(D-1)(1-2\lambda)} \\ & -\frac{(D+2)(D^3-4D^2+7D-8)}{(D-1)(1-2\lambda)^2} + \frac{(D+2)(D^2-6)}{6(1-2\alpha\lambda)} \\ & -\frac{2(D+2)(\alpha D^2-2\alpha D-D-1)}{D(1-2\alpha\lambda)^2}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} b_2(\lambda) = & \frac{(D+2)(D^3-2D^2-11D-12)}{12(D-1)(1-2\lambda)} - \frac{D^3-4D^2+7D-8}{(D-1)(1-2\lambda)^2} \\ & +\frac{(D+2)(D^2-6)}{6D(1-2\alpha\lambda)} - \frac{2(\alpha D^2-2\alpha D-D-1)}{D(1-2\alpha\lambda)^2}. \end{aligned} \tag{B.4}$$

These equations are found by using background field techniques with a metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$, so that the scale dependent action is a function of both the background metric, $\bar{g}_{\mu\nu}$, and the expectation value of the quantum fluctuations, $\bar{h}_{\mu\nu}$, i.e. $\Gamma_k[\bar{g}_{\mu\nu}, \bar{h}_{\mu\nu}]$. For the derivation and more details see references [10, 11, 14].

Vector dominance gauge

When the gauge fixing constant α goes to infinity, i.e. $\frac{1}{\alpha} = 0$, the $(1 - 2\alpha\lambda)$ terms in the flow dominate over the $(1 - 2\lambda)$ terms. Hence, as explained in the text, when $\alpha > 1$ we need to rescale the coupling constants. We get a simpler version of the β-functions in this limit because the terms with $(1-2\alpha\lambda)$ in the denominator vanish. The beta functions are found as

$$\beta_g = -2\lambda + \frac{g}{2}D(D+2)(D-5) - D(D+2)g\frac{(D-1)g + \frac{1}{D-2}(1 - 4\frac{D-1}{D}\lambda)}{2g - \frac{1}{D-2}(1-2\lambda)^2}, \tag{B.5}$$

$$\beta_\lambda = (D-2)g + \frac{(D-2)(D+2)g^2}{2(D-2)g - (1-2\lambda)^2}. \tag{B.6}$$

See [121] for a more detailed study in this limit. The analytical solutions for the rescaled fixed points are given by,

$$\tilde{\lambda} = \frac{D^2 - D - 4 - \sqrt{2D(D^2 - D - 4)}}{2(D-4)(D+1)}, \tag{B.7}$$

$$\tilde{g} = \frac{(\sqrt{D^2 - D - 4} - \sqrt{2D})^2}{2(D-4)^2(D+1)^2}. \tag{B.8}$$

B.2 The flow from bimetric truncation

A detailed study on the bimetric truncations in quantum gravity and the resulting split-symmetry can be found in [113]. In this approach, the background and the dynamical parts of the metric are treated seperately. The coefficient functions defining the flow equations for the dynamical sector with the optimised cut-off are given by

$$a_1(\lambda) = 4D + \frac{(D+1)(D+2)(D-4)}{(1-2\lambda)^2}\lambda - \frac{(D+1)(D-6)D}{2(1-2\lambda)^2}, \tag{B.9}$$

$$a_2(\lambda) = \frac{(D+1)(D-4)}{(1-2\lambda)^2}\lambda - \frac{(D+1)(D-6)D}{2(D+4)(1-2\lambda)^2}, \tag{B.10}$$

$$\begin{aligned} b_1(\lambda) = &-\frac{(D-4)(D+1)(D+2)D^2}{12(D-2)(1-2\lambda)} - \frac{8(D-1)D}{(1-2\lambda)^3} + \frac{(D+2)(D(4D-11)+3)D}{3(D-2)(1-2\lambda)^2} \\ &+\frac{2}{3}(D^2 + 2D + 12), \end{aligned} \tag{B.11}$$

$$\begin{aligned} b_2(\lambda) = &-\frac{D^5 - 15D^4 + 108D^3 + 4D(D-4)(D+1)(D+2)(D+4)\lambda^2}{12(D-2)(D+4)(1-2\lambda)^3} \\ &+\frac{D(D+4)(D^3 - 10D^2 + 15D - 10)\lambda + 95D^2 - 84D}{3(D-2)(D+4)(1-2\lambda)^3} \end{aligned} \tag{B.12}$$

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